Exercises on Companion Matrices and Difference Equations (Set 1)

1. Find the companion matrix for each polynomial below.

a. (★) $t^2 + 3t - 7$ b. $t^3 - t^2/4 + 2t - \sqrt{11}$ c. (★) $5 + 3t - 2t^2 + 13t^3 + t^4$

d. $t^6 + 1$ e. $(t^5 - 1)/(t - 1)$ f. (★) $t^5 - 6t^3 + t$

2. For each part of exercise 1, verify that the characteristic polynomial of the companion matrix you found is equal to the given polynomial. Do this by either (1) directly computing the determinant formula for characteristic polynomials or (2) entering the determinant formula at Wolfram Alpha.

3. (★) Each part below shows a matrix that we have studied earlier in the course. In each case, decide whether the matrix is a companion matrix C_p, for some polynomial $p(t)$. If it is, find $p(t)$.

a. $W = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ b. $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ c. $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (This is the Fibonacci matrix)

d. $C = \begin{bmatrix} 5 & 8 & -1 \\ -1 & 5 & 8 \\ 8 & -1 & 5 \end{bmatrix}$ (A circulant matrix) e. $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ f. $I+N = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Find the characteristic polynomial for each matrix (Answers can be checked on Wolfram Alpha)

a. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{bmatrix}$ b. $\begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} a & b & c & d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{bmatrix}$

5. Assuming that a is not zero, find the inverse for each matrix. Then conjecture an expression for the inverse of a companion matrix.

a. $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{bmatrix}$

6. (★) For each part, find the eigenvalues of the given matrix, and one nonzero eigenvector for each eigenvalue.

a. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 4 & -6 & 4 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & 3 & 3 & 2 \end{bmatrix}$

Hints:

b. The characteristic polynomial is a cubic. Try to guess one root, and use that to factor the polynomial.

c. For this characteristic polynomial, add 9 to both sides of the equation $p(t) = 0$, and notice the pattern of coefficients for $p(t) + 9$. Use the binomial theorem.

d. Remember what you learned about palindromic polynomials!

7. Follow this outline to prove that the eigenspaces of companion matrices are all one dimensional.
   a. Prove: If C_p is an $n \times n$ companion matrix, and if λ is any scalar, then the matrix $\lambda I - C_p$ has rank at least $n - 1$. [Hint: Show that the first $n - 1$ rows are linearly independent. Why does that show that the rank is at least $n - 1$?]
   b. The *Nullity Rank Theorem* from linear algebra says for an $n \times n$ matrix the dimension of the null space equals n minus the rank. Use this and part a to show that the dimension of the null space of $\lambda I - C_p$ is at most 1.
   c. Use part b to show that if λ is an eigenvalue of C_p the corresponding eigenspace has dimension exactly 1.

8. (★) A sequence is defined by the following conditions: $a_0 = 1,\ a_1 = 3,\ a_2 = -2,$ and for every $k \geq 0$, $a_{k+3} = 5a_{k+2} - 4a_{k+1} + a_k$. Express this in the form of a vector difference equation $\mathbf{a}_{k+1} = C\mathbf{a}_k$ where C is a companion matrix. Then give an equation for a_k as a function of k. That means the equation has no variables other than a_k and k. [This problem does not require you to diagonalize C.]

9. A sequence is defined by the following conditions: $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1$ and for every $k \geq 0$, $a_{k+4} = a_{k+3}/2 + a_{k+2}/3 + a_{k+1}/4 + a_k/5$. Express this in the form of a vector difference equation $\mathbf{a}_{k+1} = C\mathbf{a}_k$ where C is a companion matrix. Then give an equation for a_k as a function of k. That means the equation has no variables other than a_k and k. [This problem does not require you to diagonalize C.]

10. (★) Suppose a sequence satisfies the difference $z_{k+4} = dz_{k+3} + cz_{k+2} + bz_{k+1} + az_k$. Where a, b, c, and d are fixed constants, and $a \neq 0$. Use the difference equation to express z_k in terms of z_{k+1}, z_{k+2}, z_{k+3}, and z_{k+4}. Then use your result and the to find a matrix B for which the following equation holds

$$B\begin{bmatrix} z_{k+1} \\ z_{k+2} \\ z_{k+3} \\ z_{k+4} \end{bmatrix} = \begin{bmatrix} z_k \\ z_{k+1} \\ z_{k+2} \\ z_{k+3} \end{bmatrix}.$$

[Hint: think about B as a matrix that performs row operations on the vector it multiplies.] How does your result relate to problem 5? Explain why this makes sense.

11. For each part of this exercise use the given difference equation and initial values to work the next three terms of the sequence. (That is, terms 2, 3, and 4 for part a, terms 3, 4, and 5 for part b, etc.) Then use the methods discussed in class to find an equation for a_k as a function of k. Your final answer should not involve any matrices or vectors. (For part d, you are not required to compute $p'(\lambda)$ for all the eigenvalues.) Notice that in each case the companion matrix you formulate should be equal to one that appeared in problem 6. Finally, for parts a, b, and c, verify that your equation for a_k gives the correct values for two of the terms you already computed.
    a. $a_{k+2} = 2a_{k+1} + a_k\ ;\ a_0 = 0,\ a_1 = 1$
    b. $a_{k+3} = 3a_{k+2} - 2a_k\ ;\ a_0 = 0,\ a_1 = 0,\ a_2 = 1$
    c. $a_{k+4} = 4a_{k+3} - 6a_{k+2} + 4a_{k+1} + 8a_k\ ;\ a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 1$
    d. $a_{k+5} = 2\,a_{k+4} + 3a_{k+3} + 3a_{k+2} + 2a_{k+1} - a_k\ ;\ a_0 = a_1 = a_2 = a_3 = 0,\ a_4 = 1$

12. To extend our results on generalized Fibonacci numbers to other initial vectors, we would need to be able to solve an equation of the form $P\mathbf{x} = \mathbf{a}_0$ where P is a Vandermonde matrix, for $\mathbf{a}_0$ something other than $\mathbf{e}_n$. For example, suppose $\mathbf{a}_0 = \mathbf{e}_1$. Can you find a simple form for the solution of x in that case?

Selected Solutions: Exercises on Companion Matrices and Difference Equations (Set 1)

1. a. $C=\begin{bmatrix}0 & 1\\ 7 & -3\end{bmatrix}$. c. $C=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -5 & -3 & 2 & -13\end{bmatrix}$. f. $C=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & 6 & 0\end{bmatrix}$.

3. The matrices in parts a, c, and e are companion matrices. Their respective characteristic polynomials are t^4-1, t^2-t-1, and t^4.

6. a. Eigenvalues are $1\pm\sqrt{2}$. Corresponding nonzero eigenvectors are $\begin{bmatrix}1 & 1\pm\sqrt{2}\end{bmatrix}^T$
   b. Eigenvalues are 1 and $1\pm\sqrt{3}$. Corresponding nonzero eigenvectors are $[1\quad 1\quad 1]^T$ and $\begin{bmatrix}1 & 1\pm\sqrt{3} & 4\pm 2\sqrt{3}\end{bmatrix}^T$.
   c. For this problem, it helps to recognize that the characteristic polynomial is $p(t)=(t-1)^4-9$. That is a difference of two squares, and so factors as
   $$((t-1)^2-3)((t-1)^2+3)=(t^2-2t-2)(t^2-2t+4).$$
   Eigenvalues are $1\pm\sqrt{3}$ and $1\pm i\sqrt{3}$. Corresponding eigenvectors are given by $\begin{bmatrix}1 & 1\pm\sqrt{3} & 4\pm 2\sqrt{3} & 10\pm 6\sqrt{3}\end{bmatrix}^T$ and $\begin{bmatrix}1 & 1\pm i\sqrt{3} & -2\pm 2i\sqrt{3} & -8\end{bmatrix}^T$. Notice that the nonreal eigenvalues equal 2α where α is a sixth root of unity. In particular, $\alpha^3=-1$, accounting for the fact that the cube of each of these eigenvalues equals -8.
   d. Here the characteristic polynomial is a palindromial of odd degree, and hence has one root equal to -1. That means we can factor out $(t+1)$ to express the characteristic polynomial as $p(t)=(t+1)(t^4-3t^3-3t+1)$. Now the quartic factor has no roots of 1 or -1, and that shows that the roots come in reciprocal pairs. Pursuing this idea leads to the factorization $p(t)=(t+1)(t^2-\frac{3+\sqrt{17}}{4}t+1)(t^2-\frac{3-\sqrt{17}}{4}t+1)$, and hence to the roots: -1, $\frac{3+\sqrt{17}\pm\sqrt{10+6\sqrt{17}}}{4}$, and $\frac{3-\sqrt{17}\pm i\sqrt{6\sqrt{17}-10}}{4}$. As usual, eigenvectors can be expressed in the form $[1\quad \lambda\quad \lambda^2\quad \lambda^3\quad \lambda^4]^T$ where λ is the corresponding eigenvalue, but actually working out and simplifying the powers of the eigenvalues (other than -1) is too hard and not very illuminating.

8. The vector difference equation and the solution equation are
$$\begin{bmatrix}a_{k+1}\\ a_{k+2}\\ a_{k+3}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -4 & 5\end{bmatrix}\begin{bmatrix}a_k\\ a_{k+1}\\ a_{k+2}\end{bmatrix} \quad\text{and}\quad a_k=[1\ \ 0\ \ 0]\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -4 & 5\end{bmatrix}^k\begin{bmatrix}1\\ 3\\ -2\end{bmatrix}.$$

10. From $z_{k+4}=dz_{k+3}+cz_{k+2}+bz_{k+1}+az_k$ we find $z_k=\frac{1}{a}z_{k+4}-\frac{d}{a}z_{k+3}-\frac{c}{a}z_{k+2}-\frac{b}{a}z_{k+1}$. This allows us to write
$$\begin{bmatrix}z_k\\ z_{k+1}\\ z_{k+2}\\ z_{k+3}\end{bmatrix}=\begin{bmatrix}\frac{1}{a}z_{k+4}-\frac{d}{a}z_{k+3}-\frac{c}{a}z_{k+2}-\frac{b}{a}z_{k+1}\\ z_{k+1}\\ z_{k+2}\\ z_{k+3}\end{bmatrix}=\begin{bmatrix}\frac{-b}{a} & \frac{-c}{a} & \frac{-d}{a} & \frac{1}{a}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}z_{k+1}\\ z_{k+2}\\ z_{k+3}\\ z_{k+4}\end{bmatrix}$$
This is consistent with the results from problem 5. Indeed, the vector version of the difference equation is $\mathbf{z}_{k+1}=C_p\mathbf{z}_k$, where C_p is a companion matrix. If it is invertible, we see that $\mathbf{z}_k=C_p^{-1}\mathbf{z}_{k+1}$. Therefore, it is not surprising that the matrix B in this problem turns out to be C_p^{-1}.