Lecture 4 Exercises

1. (★) Let D be a diagonal matrix with diagonal entries $d_1, d_2, d_3, \cdots, d_n$. Let p be any polynomial. Find a matrix representation for $p(D)$.

2. (★) Let N be the 4 x 4 matrix with 1's on the superdiagonal and 0's everywhere else. Let $p(x) = 17x^9 + 3x^4 - 7x^3 - 4x^2 + 2x + 6$. Compute the following:
   a. $p(N)$ c. $p(N) + p(N^T)$
   b. $p(N^T)$ d. $p(N + N^T)$ [Hint: use the formula for M^k in lecture note 4biii.]

3. (★) For the matrix N in problem 2, let M be the matrix $N + N^T$. Also let $p(t) = t^4 - 3t^2 + 1$ and $f(t) = t^6 + 2t^5 - 3t^4 - 6t^3 + t^2 + 5t - 1$.
   a. Show that $p(M) = 0$.
   b. Use polynomial long division to find the remainder when $f(t)$ is divided by $p(t)$.
   c. Use your result from part b to compute $f(M)$.

4. Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
   a. Study the powers of M and look for a pattern.
   b. Find a formula for M^k for positive integers k, and prove your formula using induction.
   c. Prove the following: $F_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ where F_k is the kth Fibonacci number.
   d. Find a polynomial $p(t)$ for which M is the companion matrix.

5. (★) Here is one way to find the minimal polynomial for a matrix M. Assuming that M is not a scalar multiple of the identity matrix, compute M^2 and try to express that as a linear combination of the form $aM + bI$. If you succeed, you will have found that $M^2 - aM - bI = 0$, and therefore $p(t) = t^2 - at - b$ is the minimal polynomial of M (why?). Otherwise, $p(M)$ cannot equal zero for any quadratic polynomial. So compute M^3 and try to express *it* as a linear combination of the form $aM^2 + bM + c$. If you succeed, you will have found a polynomial of the form $t^3 - at^2 - bt - c$ which is the minimal polynomial (why?). Continue in this way until you find a polynomial of minimal degree for which substituting M produces the zero matrix.
   Use this method to find the minimal polynomials of the matrix M in problem 3, and for N and $N + N^T$ where N is the matrix in problem 1.

6. In class we explored patterns in powers of a square matrix of the form $N + N^T$, where N has the same pattern as in problem 1. Perform a similar exploration for $N - N^T$ in the case of 2x2, 3x3, and 4x4 versions of N.

7. Let W be an nxn matrix obtained from the identity matrix by moving the first row below the last the row. For example, the 3x3 version of W is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.
   a. Explore patterns in the powers of W for n = 3, 4, and 5. Conjecture a general pattern for powers of W for any n.
   b. Use your conjecture to compute $p(W)$ where p is any polynomial of degree n-1 or less. [Such a matrix is called a *circulant* matrix.]
   c. Use your conjecture to find the minimal polynomial p for W. If possible prove the minimal condition holds – namely that for any polynomial q of lower degree than p, it is impossible for $q(W)$ to equal the zero matrix.

8. Let X be the nxn with 1's on both diagonals: from the 11 to the nn position, and from the 1n to n1 position. Explore the powers of X. Conjecture a formula for X^n and if possible prove your conjecture.

Answers and Hints to selected Lecture 4 Exercises

1. With p and D as in the problem statement, $p(D) = \begin{bmatrix} p(d_1) & & & \\ & p(d_2) & & \\ & & \ddots & \\ & & & p(d_n) \end{bmatrix}$. That is, $p(D)$ is a diagonal matrix whose diagonal entries are found by applying p to the diagonal entries of D. To prove this, note that $D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}$ (as can be proved by induction). Therefore, if

$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$, we find

$$p(D) = a_m D^m + a_{m-1}D^{m-1} + \cdots + a_1 D + a_0 I$$

$$= a_m \begin{bmatrix} d_1^m & & & \\ & d_2^m & & \\ & & \ddots & \\ & & & d_n^m \end{bmatrix} + a_{m-1} \begin{bmatrix} d_1^{m-1} & & & \\ & d_2^{m-1} & & \\ & & \ddots & \\ & & & d_n^{m-1} \end{bmatrix} + \cdots + a_1 \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} + a_0 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_m d_1^m + a_{m-1}d_1^{m-1} + \cdots + a_0 & & & \\ & a_m d_2^m + a_{m-1}d_2^{m-1} + \cdots + a_0 & & \\ & & \ddots & \\ & & & a_m d_n^m + a_{m-1}d_n^{m-1} + \cdots + a_0 \end{bmatrix}$$

$$= \begin{bmatrix} p(d_1) & & & \\ & p(d_2) & & \\ & & \ddots & \\ & & & p(d_n) \end{bmatrix}.$$

2. (a) We know $N^0 = I$, $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and all higher powers of N are zero matrices. Therefore,

$$p(N) = 17N^9 + 3N^4 - 7N^3 - 4N^2 + 2N + 6I = -7N^3 - 4N^2 + 2N + 6I = \begin{bmatrix} 6 & 2 & -4 & -7 \\ 0 & 6 & 2 & -4 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

(b) We can proceed as in part (a). Or we can recognize that properties of the transpose include

- $(rA)^T = r \cdot A^T$
- $(A+B)^T = A^T + B^T$
- $(A^k)^T = (A^T)^k$.

Together, these imply that, for any polynomial p and any square matrix A, $p(A^T) = (p(A))^T$.
Accordingly,

$$p(N^T) = (p(N))^T = \begin{bmatrix} 6 & 2 & -4 & -7 \\ 0 & 6 & 2 & -4 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}^T = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ -4 & 2 & 6 & 0 \\ -7 & -4 & 2 & 6 \end{bmatrix}.$$

(c) Combining the results of (a) and (b) we find that $p(N) + p(N^T) = \begin{bmatrix} 12 & 2 & -4 & -7 \\ 2 & 12 & 2 & -4 \\ -4 & 2 & 12 & 2 \\ -7 & -4 & 2 & 12 \end{bmatrix}$.

(d) Let $M = N + N^T$. We have seen in class that $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $M^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$,

$M^3 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$, and $M^4 = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}$. For $n = 9$, which is odd, we use the formula

$M^9 = \begin{bmatrix} 0 & F_9 & 0 & F_8 \\ F_9 & 0 & F_{10} & 0 \\ 0 & F_{10} & 0 & F_9 \\ F_8 & 0 & F_9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 34 & 0 & 21 \\ 34 & 0 & 55 & 0 \\ 0 & 55 & 0 & 34 \\ 21 & 0 & 34 & 0 \end{bmatrix}$. Combining all these matrices, we find that

$p(M) = 17M^9 + 3M^4 - 7M^3 - 4M^2 + 2M + 6I = \begin{bmatrix} 8 & 566 & 5 & 350 \\ 566 & 13 & 916 & 5 \\ 5 & 916 & 13 & 566 \\ 350 & 5 & 566 & 8 \end{bmatrix}$.

Note that $p(N + N^T) \neq p(N) + p(N^T)$.

3. (a) Viewing the matrices in the answer to 2d, we see that $M^4 + I = 3M^2$. This shows that $p(M) = 0$, as required. (This result can also be verified directly in Freemat.)
(b) Long division shows that $f(t) = (t^2 + 2t)p(t) + 3t - 1$, so the remainder is $r(t) = 3t - 1$.
(c) We use the fact that $f(M) = (M^2 + 2M)p(M) + r(M) = r(M) = 3M - I$. Therefore,
$f(M) = \begin{bmatrix} -1 & 3 & 0 & 0 \\ 3 & -1 & 3 & 0 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 3 & -1 \end{bmatrix}$. **[NOTE:** this is much easier than computing $f(M)$ as we did in problem 2d! **]**

5. Here is the solution for the matrix $N + N^T$. Let $M = N + N^T$, and use the facts that

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad M^3 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \quad \text{and } M^4 = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$

Clearly M is not a multiple of the identity matrix, so $p(M)$ is not zero for any linear polynomial p. Any combination of M and I will have a 0 in the 1,3 position, and so cannot equal M^2. This shows that no combination of I, M, and M^2 can equal zero, so $p(M)$ is not zero for any quadratic polynomial p. Similarly, no combination of I, M, and M^2 can have a 1 in the 1,4 position, so no such combination can equal M^3. Therefore, the minimal polynomial must have degree at least 4. Now M^4 has nonzero entries only on the main diagonal, and on the 2nd super and sub diagonals. Since M and M^3 cannot contribute anything to those entries, we try to express M^4 as a combination

$$aI + bM^2 = \begin{bmatrix} a+b & 0 & b & 0 \\ 0 & a+2b & 0 & b \\ b & 0 & a+2b & 0 \\ 0 & b & 0 & a+b \end{bmatrix}.$$

Inspecting the first row we see that this combination can only equal M^4 if $a+b = 2$ and $b = 3$. So the only possibility is with $a = -1$ and $b = 3$. And this combination does indeed work. That is, $M^4 = -I + 3M^2$, showing that $M^4 - 3M^2 + I = 0$. Therefore the minimal polynomial for M is $t^4 - 3t^2 + 1$.