Exercises on Lag Operator and Difference Equations

1. This is a repeat of exercise 11 of **Companion Matrices Exercise Set 1**. Do this exercise if you have not already done so.

   For each part of this exercise use the given difference equation and initial values to work out the next three terms of the sequence. (That is, terms 2, 3, and 4 for part a, terms 3, 4, and 5 for part b, etc.) Then use the methods discussed in class to find an equation for a_k as a function of *k*. Your final answer should not involve any matrices or vectors. (For part d, you are not required to compute $p'(\lambda)$ for all the eigenvalues.) Notice that in each case the companion matrix you formulate should be equal to one that appeared in problem 6. Finally, for parts a, b, and c, verify that your equation for a_k gives the correct values for two of the terms you already computed.

   a. $a_{k+2} = 2a_{k+1} + a_k$; $a_0 = 0,\ a_1 = 1$
   b. (★) $a_{k+3} = 3a_{k+2} - 2a_k$; $a_0 = 0,\ a_1 = 0,\ a_2 = 1$
   c. $a_{k+4} = 4a_{k+3} - 6a_{k+2} + 4a_{k+1} + 8a_k$; $a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 1$
   d. $a_{k+5} = 2\,a_{k+4} + 3a_{k+3} + 3a_{k+2} + 2a_{k+1} - a_k$; $a_0 = a_1 = a_2 = a_3 = 0,\ a_4 = 1$

2. Express each difference equation in problem 1 in the form $p(L)\{a_k\} = 0$ giving $p(L)$ as an explicit polynomial in *L*. (★ Solution given for part b.)

3. If *V* and *W* are vector spaces, a linear transformation from *V* to *W* is a function $T: V \to W$ for which the following two algebraic identities hold for all scalars *c* and all vectors **u** and **v** in *V*:

   (i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
   (ii) $T(c\mathbf{u}) = c\,T(\mathbf{u})$.

   In linear algebra we focus primarily on the case that *V* and *W* are $\mathbb{R}^m$ and $\mathbb{R}^n$ (or respective subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$). In this case every linear transformation is the same as multiplying by a particular constant matrix, and the identities (i) and (ii) are familiar properties of matrix and vector arithmetic. But as discussed in the lecture, we can also consider *V* and *W* to be $\mathbb{R}^\infty$ (or $\mathbb{C}^\infty$), and vectors are given by infinite sequences of real (or complex) numbers. In that context, we can ask whether a particular function from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ (or from $\mathbb{C}^\infty$ to $\mathbb{C}^\infty$) is a linear transformation, meaning that it obeys (i) and (ii). The following questions illustrate this idea.

   a. Show that the left shift operator *L* is a linear transformation from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$.
   b. Show that L^m is a linear transformation from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ for any positive integer *m*. [Hint: use induction.]
   c. Show that for any scalar *c*, cL^m is a linear transformation from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ for any positive integer *m*.
   d. Show that for any polynomial $p(t)$ with real coefficients, $p(L)$ is a linear transformation from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$.
   e. (★) We can define eigenvalues and eigenvectors for the transformation *L* analogously to how we define them for matrices: If there is a nonzero **u** in $\mathbb{R}^\infty$ such that $L\mathbf{u} = \lambda\mathbf{u}$ for some scalar λ, then we say that λ is an eigenvalue of *L*; if λ is an eigenvalue of *L* and if **v** is any element of $\mathbb{R}^\infty$ for which $L\mathbf{v} = \lambda\mathbf{v}$, we say that **v** is an eigenvector of *L* corresponding to the eigenvalue λ. Show that every real number is an eigenvalue of *L* and find one nonzero eigenvector.

4. (★) A sequence is defined by the conditions $a_0 = 17,\ a_1 = 5,$ and for $k \geq 0$ $a_{k+2} = a_{k+1} + a_k$. Find an equation for a_k as a function of *k*.

5. In problem 1b you were asked to find an equation for the sequence defined by

$$a_{k+3} = 3a_{k+2} - 2a_k\ ;\ \ a_0 = 0,\ a_1 = 0,\ a_2 = 1.$$

For this problem, we will refer to that sequence as $\{B_k\}_{k=0}^{\infty}$ = (0,0,1,3,9,25,69,…). Now you are asked to find a solution to the same difference equation that has different initial terms:

$$a_{k+3} = 3a_{k+2} - 2a_k\ ;\ \ a_0 = 3,\ a_1 = 1,\ a_2 = 5.$$

To do so, note that $\{B_k\}$, $L\{B_k\}$, and $L^2\{B_k\}$ are solutions of the same difference equation, as is any linear combination of those three sequences. Accordingly, find constants *u*, *v*, and *w* such that the sequence $u\{B_k\} + vL\{B_k\} + wL^2\{B_k\}$ has initial terms 3, 1, and 5. Note that gives our desired equation for a_k as $a_k = uB_k + vB_{k+1} + wB_{k+2}$.

Solutions to Selected Exercises on Lag Operator and Difference Equations

1. b. The initial terms are $a_0 = 0,\ a_1 = 0$, and $a_2 = 1$. Taking $k = 0$ in the difference equations gives $a_3 = 3a_2 - 2a_0 = 3$. Then with $k = 1$ we have $a_4 = 3a_3 - 2a_1 = 9$, and with $k = 2$ we have $a_5 = 3a_4 - 2a_2 = 27 - 2 = 25$.

   The difference equation $a_{k+3} = 3a_{k+2} - 2a_k$ is expressed in vector-matrix form as $\mathbf{v}_{k+1} = C\mathbf{v}_k$, where $\mathbf{v}_k = [a_k \quad a_{k+1} \quad a_{k+2} \quad \cdots \quad a_{k+n-1}]^T$ and C is the companion matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix}$. This has characteristic polynomial $p(t) = t^3 - 3t^2 + 2$. By inspection we notice that $p(1) = 0$, giving us one root and leading to the factorization $p(t) = (t-1)(t^2 - 2t - 2)$. Applying the quadratic formula to the second factor reveals roots of $1 \pm \sqrt{3}$. Thus, the eigenvalues of C are 1, $1+\sqrt{3}$, and $1-\sqrt{3}$. Notice also that these are distinct. Therefore, we can apply the result from class giving the solution of a difference equation with initial vector $\mathbf{e}_n$ as

$$a_k = \frac{\lambda_1^k}{p'(\lambda_1)} + \frac{\lambda_2^k}{p'(\lambda_2)} + \frac{\lambda_3^k}{p'(\lambda_3)}$$

   where the λ's are 1, $1+\sqrt{3}$, and $1-\sqrt{3}$. We differentiate the characteristic polynomial to find $p'(t) = 3t^2 - 6t = 3t(t-2)$. This leads to $p'(1) = -3$, $p'\left(1+\sqrt{3}\right) = 3\left(1+\sqrt{3}\right)\left(-1+\sqrt{3}\right) = 6$, and $p'\left(1-\sqrt{3}\right) = 3\left(1-\sqrt{3}\right)\left(-1-\sqrt{3}\right) = -3\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right) = 6$. Therefore

$$a_k = -\frac{1}{3} + \frac{\left(1+\sqrt{3}\right)^k}{6} + \frac{\left(1-\sqrt{3}\right)^k}{6}$$

2. b. Based on the definition of the lag operator L, we can consider $a_{k+3} = L^3\, a_k$ and $a_{k+2} = L^2\, a_k$. Therefore the difference equation $a_{k+3} = 3a_{k+2} - 2a_k$ can be written $L^3\, a_k = 3L^2\, a_k - 2a_k$, or equivalently, $L^3\, a_k - 3L^2\, a_k + 2a_k = 0$. This is the same as $(L^3 - 3L^2 + 2)a_k = 0$, so the polynomial in this problem is $p(t) = t^3 - 3t^2 + 2$.

3. e. Let c be any scalar, and let $\mathbf{v}$ be the sequence $(a_0,\ a_1,\ a_2,\ \cdots)$. Then $L\mathbf{v} = c\mathbf{v}$ if and only if $(a_1,\ a_2,\ a_3,\ \cdots) = c(a_0,\ a_1,\ a_2,\ \cdots) = (ca_0,\ ca_1,\ ca_2,\ \cdots)$. This equation holds if and only if $a_1 = ca_0,\ a_2 = ca_1 = c \cdot ca_0 = c^2a_0,\ a_3 = ca_2 = c \cdot c^2a_0 = c^3a_0$, and so on. We recognize that this gives $\mathbf{v}$ as $(a_0,\ ca_0,\ c^2a_0,\ c^3a_0,\ \cdots) = a_0(1,\ c,\ c^2,\ c^3,\ \cdots)$. In particular, every scalar **c** is an eigenvalue of L, with corresponding nonzero eigenvector $(1,\ c,\ c^2,\ c^3,\ \cdots)$.

4. The desired sequence is defined by the Fibonacci difference equation, but it has initial terms 17 and 5 rather than 0 and 1. However, we can find such a sequence as a combination of $\{F_k\}_{k=0}^{\infty} =$ (0,1,1,2,3,5,8,…) and $\{G_k\}_{k=0}^{\infty} =$ (1,0,1,1,2,3,5,8,…). In fact, if $\{a_k\}_{k=0}^{\infty} = 17\{G_k\}_{k=0}^{\infty} + 5\{F_k\}_{k=0}^{\infty}$, then we can see by inspection that $a_0 = 17$ and $a_1 = 5$, and remaining a's obey the Fibonacci difference equation. In particular, we see that $a_k = 17G_k + 5F_k = 17F_{k-1} + 5F_k$. This is valid for all natural numbers k if we define $F_{-1} = 1$. Thus we have expressed a_k as a function of k using Fibonacci numbers. To express it explicitly in terms of k, we can use our known formula for the k^{th} Fibonacci number $F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$. This leads to

$$a_k = \frac{17}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] + \frac{5}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right].$$

   This can be simplified, but as it stands it does give a_k as a function of k.