Harvey Plotter

and the Circle of Irrationality

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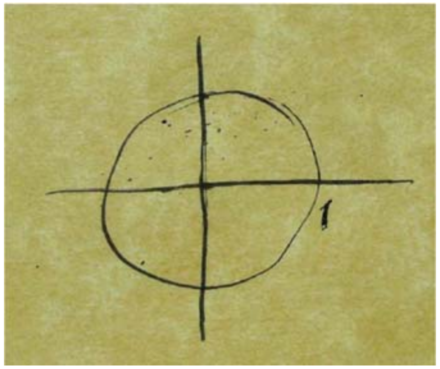
Harvey Plotter woke with a groan and clutched his forehead. Searing pain screamed along his radical scar. With great effort, he broke the mental connection with Lord Voldemorphism, who had given him that scar so many years ago.

"Harvey? Are you alright?" asked Hymernie. "It's your scar again, isn't it?"

Harvey unclenched his hands as the pain slowly eased and the fog in his brain lifted. "Yeah," he said. "You-know-who is really upset about something."

Rong joined them at the chairs round the fire in the Graphindor common room, looking worried. "Why? What did you see?"

"It was some kind of cryptic diagram." He unrolled a scroll of parchment and sketched the image that still burned in his memory.



"What does it mean?" said Rong.

Hymernie shot him a disapproving look. "Honestly, Rong, don't you ever read your textbooks? That's clearly the circularum unitatus."

Rong shuffled his feet. "It just looks like the unit circle to me."

Hymernie rolled her eyes, but before she could reply, Harvey cut in. "Never mind that; we don't have time. You-know-who is trying to find all the rational points on that circle."

"Rational points?" said Rong.

"Yeah, you know, points like (1,0) and (0,1) where the coordinates are both rational numbers."

"But why does You-know-who want to find them?"

"I don't know, but if he does, they must be important. We have to find out why. And we have to find the points before he does."

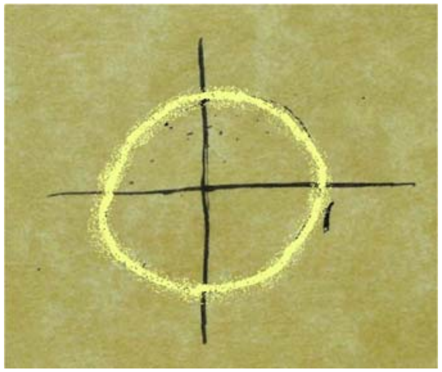
"Well, let's get going, mate," said Rong. "You already found two, (1,0) and (0,1), and by symmetry, (-1,0), and (0,-1) are two more. That's four down. How many others could there be?"

"That's just it, Rong," Hymernie said shaking her head. "Even if we find more, how can we be sure we have them all?"

"What about magic?" asked Harvey. "Isn't there some spell you can use?"

"Maybe," Hymernie said, "but wouldn't You-know-who have tried that straight away? I guess it wouldn't hurt to try, though." She raised her wand. "Rationalus revealious!"

All round the circle, points of light began to appear, increasing in number and intensity until the entire circumference glowed brightly.



Rong frowned at the parchment. “Looks to me like every bloody point on the circle is rational!”

“No,” said Hymernie. “There are loads of non-rational points, like

$$\left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right) \text{ and } \left(\frac{1}{3}, \sqrt{\frac{8}{9}}\right).$$

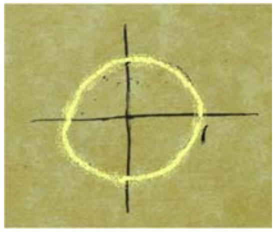
In fact, for any x between -1 and 1, the unit circle equation

$$x^2 + y^2 = 1$$

gives

$$y = \pm\sqrt{1 - x^2}.$$

But if x is a simple fraction, the y’s are almost always irrational. No, I think what the spell must be showing is that the rational points are packed so closely together that it’s impossible to distinguish them.”



“Is You-know-who looking for irrational points, too?” Rong asked Harvey.

“No,” said Harvey with certainty. “They would be useless to him.”

Applied numerology was considered a dark art, but Harvey's connection to Voldemorphism gave him a gut instinct for it that his friends trusted.

Rong became unusually thoughtful. "You know, a line between two rational points always has a rational slope."

"How's that?" asked Harvey.

"Well, like, (0,1) and (3/5, 4/5) are rational points. The slope between them is

$$\frac{\frac{4}{5}-1}{\frac{3}{5}-0}.$$

All the numbers in there are rational, so when you work out the result, it's sure to be rational, too."

Hymernie seemed skeptical. "Okay, but where does that get us?"

"Well," Rong continued uncertainly. "Maybe if you draw a line with rational slope through one point, like (0,-1), it will intersect the circle in another rational point. Then, by drawing lots of lines with rational slopes, you could find lots of rational points."

Hymernie sighed. "Rong, you're confusing the converse and the contrapositive. It's a classic fallacy."

Rong blinked, not willing to admit that he needed an explanation.

Harvey chimed in to help out. "You mean just because slopes between rational points are rational, that doesn't mean it works the other way round. Just starting with a rational slope from a rational point doesn't mean we'll hit the circle at another rational point."

“Exactly,” Hymernie replied.

“But Rong’s idea is still a good one. If we *could* generate new rational points his way, the fact he gave us guarantees that we’d be able to find them *all* that way. There would be exactly one for every rational slope.”

Rong nodded quickly. “Yeah, that’s exactly what I meant. Thanks, mate.”

“So let’s have a go at sorting this out,” said Harvey. “We may not have much time. Does drawing lines with rational slopes always lead to points on the circle that are *rational?* Let’s try a few examples.”

He reached for the quill, but Hymernie was there first. Her eyes had lit up the moment he stated an unproven conjecture.

The quill flew over the parchment as she spoke. “Start with a line through (0,-1) with a slope of 1/2. Its y intercept will be -1, so its equation is $y = \frac{1}{2}x - 1$. It generates a new point of intersection at the simultaneous solution of the line and circle equations.”

$$x^2 + y^2 = 1$$
$$y = \frac{1}{2}x - 1.$$

As she began to solve, Rong leaned closer and pointed to her first step,

$$x^2 + \left(\frac{1}{2}x - 1\right)^2 = 1 \qquad (1)$$

“How’d you get that?” he asked.

She turned her head towards his and for a moment that seemed to last arbitrarily long, their eyes met, and noses nearly touched.

But Harvey broke in. “Rong, Hymernie. Focus! We have to solve this before Voldemorphism does. Remember what happens to everything he gets his hands on?”

“Yes,” breathed Hymernie, breaking away from Rong’s gaze. “He…he maps everything to…”

“To evil,” Harvey finished. “And there’s no inverting it.”

The quill wavered in Hymernie’s hand. She hadn’t felt this uncertain about a math problem since taking the O.D.E. O.W.L.

“I think,” said Harvey, “you were about to say you used substitution. Right?”

$$x^2 + \left(\tfrac{1}{2}x - 1\right)^2 = 1. \qquad (1)$$

“Yes of course,” she replied, coming to herself. “And equation (1) is just a quadratic, so we can expand it and apply the quadratic formula. But that just gives us a formula to *find* the points where the line intersects the circle. It doesn’t say they’re rational. There’s a radical in the quadratic formula, you know, and that is generally a sign of irrationality. ” Involuntarily, she glanced at the scar on Harvey's forehead.

Rong said, “But we don’t need both roots of the quadratic. We already know that (0, -1) is one of them, and we just want the other one.”

Hymernie’s gasped, her eyes nearly as wide as the circle on the parchment. **“Rong, that’s it!”** she shrieked.

She embraced him with such force that the two tumbled to the Graphindor common room floor.

Rong extricated himself with an injured look. “Why are you always so surprised when I have a good idea?”

Hymernie returned to her chair with a scarlet face as Harvey cut in again. “Remember—we’re kind of in a hurry here! What did you just realize, Hymernie?”

Hymernie took a deep breath. “The roots of the quadratic equation Rong was er…interested in”—she blushed again—“are the x coordinates of the two points where the line intersects the circle. But we already know that one of those points has x = 0.   That means 0 is a root and the quadratic has to factor in the form $ax(x-b)$. And since *that* process *doesn’t* involve taking a square root, the other root will also be rational. We’ve proven Rong’s conjecture!”

“Brilliant!” said Harvey. “And that means that we can find all of the rational points by intersecting the circle with lines of rational slope through (0,-1)!”

“I’ll do it, mate,” said Rong. He adopted an uncharacteristically formal, lecturing tone.

"After simplifying the equation, $x^2 + \left(\frac{1}{2}x - 1\right)^2 = 1$ we get

$$x\left(\frac{5}{4}x - 1\right) = 0,$$

whose nonzero solution is x = 4/5. The equation of the line,

$$y = \frac{1}{2}x - 1,$$

yields y = -3/5. So we have the rational point (4/5,-3/5). And by symmetry again we find (4/5, 3/5), and the other variations possible by choosing plus or minus signs."

He tossed down the quill with a smile, clearly pleased with himself, and gazed in Hymernie's direction. His smile faded when he got only a quirked eyebrow in return. "What? Wasn't that right?"

Harvey smiled. "It's right, but I think you were getting along better before you started showing off." Rong sighed.

"But it's cool how the math worked out," Harvey continued, "so let me try one. If I take a slope of 5, then the line will be $y = 5x - 1$. Substituting that in the circle equation gives

$$x^2 + (5x - 1)^2 = 1$$

and we can simplify to find

$$26x^2 - 10x = 0.$$

I can factor that as $2x(13x - 5) = 0$, so x = 0 or 5/13, and we're interested in the 5/13. Putting that into the line equation $y = 5x - 1$ gives y = 12/13. Great! We have another rational point on the circle, (5/13, 12/13)."

“But how will we ever work out *all* the rational points?” said Hymernie. “There will be a different one for every rational slope, and there are infinitely many slopes.”

For a long time after this remark, all three friends stared silently at the parchment. Despair seemed to descend as the staggering concept of an infinitely long algebra assignment slowly sank in.

A kindly voice from behind them broke the silence. “You seem to be forgetting one of the most powerful and ancient forms of magic,” it said. The three turned to see, staring out of a portrait on the wall behind them, the piercing blue eyes of Alphas Jumblemore.

“What magic is that, professor?” said Hymernie.

“Why the magic of algebra,” he said. “Do the same steps as before, but instead of choosing a specific number for the slope, try using a generic rational number, say p/q.”

“I didn’t know we could do that,” whispered Rong.

“Let’s try,” said Harvey. “If the slope is p/q then the line has equation

$$y = \frac{p}{q}x - 1.$$

.

Substituting in the circle equation leads to

$$x^2 + \left(\frac{p}{q}x - 1\right)^2 = 1,$$

which we can expand to

$$x^2 + \frac{p^2}{q^2}x^2 - \frac{2p}{q}x + 1 = 1.$$

Now combine terms like this:

$$\left(1 + \frac{p^2}{q^2}\right)x^2 - \frac{2p}{q}x = 0."$$

He regarded the result.

$$\left(1 + \frac{p^2}{q^2}\right)x^2 - \frac{2p}{q}x = 0$$

"That's pretty ugly, but I can still factor,

$$x\left[\left(1 + \frac{p^2}{q^2}\right)x - \frac{2p}{q}\right] = 0$$

and so to find the nonzero root we have to solve

$$\left(1 + \frac{p^2}{q^2}\right)x - \frac{2p}{q} = 0."$$

Harvey paused, not as pleased with the results as he had hoped to be. "Boy, that seems pretty complicated."

$$\left(1+\frac{p^2}{q^2}\right)x-\frac{2p}{q}=0."$$

“Try multiplying through by q^2,” said Hymernie. “That will eliminate the fractions. Then you have

$$(q^2+p^2)x-2pq=0,$$

hence

$$(q^2+p^2)x=2pq$$

and

$$x=\frac{2pq}{q^2+p^2}.$$

That gives us x. Then, from the line equation,

$$y=\frac{p}{q}x-1=\frac{p(2pq)}{q(q^2+p^2)}-1=\frac{2p^2}{(q^2+p^2)}-1=\frac{p^2-q^2}{p^2+q^2}.$$

And that gives us y.

“We’ve done it! We’ve found a formula that generates all the rational points on the unit circle! For any integers p and $q\neq 0$, we get the rational point $\left(\frac{2pq}{p^2+q^2},\frac{p^2-q^2}{p^2+q^2}\right)$, and that accounts for *all* the rational points. No wait – there is also the point (0,1), which cannot be expressed in the standard form for any p and q.”

“Hymernie, you are amazing,” said Harvey. “I just wish we knew why You-know-who wants to figure this out.”

Rong hadn’t spoken since they started following Jumblemore’s advice. Looking over, Harvey saw that Rong had a faraway look, staring at the NumerologyCounts banners that hung above the fireplace. “Rong? Are you all right?”

“Huh? What?” Rong started. “Oh, yeah. I was just thinking. Those rational points we found…Do they remind you of anything?”

“What do you mean?” said Harvey.

“Well, we found the points (4/5, 3/5) and (5/13, 12/13). If you just pay attention to the whole numbers in them, you get 4, 3, 5, in the first one and 5, 12, 13, in the second. They seem familiar.”

“Rong!” cried Hymernie, shaking him by the robe. “That’s brilliant! They’re what muggles call Pythagorean triples!” She leaned closer warmly, but received only a disappointed look in response.

“I mean to say,” she continued more calmly, “that of course you figured it out, because you’re…sometimes very…thoughtful.”

“Yes,” Rong replied with a small smile of satisfaction. “I’m quite a thoughtful bloke sometimes.” He put his arm round Hymernie and turned to Harvey. Tapping a finger to his temple, he repeated, “Thoughtful.”

“But I still don’t understand. What’s a muggle triple, or whatever you said?”

Hymernie returned to the parchment. “Those rational points give proportions of *right triangles*. And in the very first lecture of Mystical Numerology, we learned that—”

“—right triangles have arbitrarily large amounts of magical power,” Harvey finished for her. He could barely remember yesterday’s lecture, let alone the first day, but something deep inside him knew this was as true as if he had proven it in a formal system. “But I still don’t understand about the triples.”

Hymernie returned to the parchment. "Look, (3/5, 4/5) goes with a 3, 4, 5 right triangle, and (5/13, 12/13) goes with a 5, 12, 13 right triangle. In the same way, our generic point

$(2pq/(p^2+q^2), (p^2-q^2)/(p^2+q^2))$ goes with a $2pq$, p^2-q^2, p^2+q^2 right triangle. We have a formula that can generate *every* triple of integers that can form the sides of a right triangle."

Harvey beamed at his two best friends, in a good mood for the first time since he'd seen the circularum unitatis in his dream. "And that means that no matter what right triangle spells You-know-who throws at us, we'll know how to fight back!"

From his portrait, Alphas Jumblemore quietly smiled down on his three young protégés. "A powerful magical triple, eh? I believe I see one of those myself!"

Definition 1. Given a curve C in the plane, a point $(x, y) \in C$ is called a rational point if and only if x and y are both rational numbers.

Exercise 1. Find (with proof) a point on the unit circle that is not a rational point, and a point that is a rational point.

Exercise 2. Exercise 2.8. Show that: Adding two rational numbers always results in a rational number. Subtracting two rational numbers always results in a rational number. Multiplying two rational numbers always results in a rational number. Under what conditions does dividing two rational numbers result in a rational number?

Exercise 3. If (x, y) and (u, v) are rational points, show that the slope of the line between them is either rational or undefined.

Exercise 4. Prove the following proposition: The point (r, s) is a rational point on the unit circle with $r \neq 0$ iff $r = 2pq/(p^2+q^2)$ and $s = (p^2 - q^2)/(p^2+q^2)$ for some relatively prime integers p and $q \neq 0$.

Definition 2. A Pythagorean triple is an ordered triple of integers (a, b, c) for which $a^2 + b^2 = c^2$. It is nontrivial if $(a, b, c) \neq (0, 0, 0)$.

Exercise 5. Prove the following proposition: The integer triple (a, b, c) is a nontrivial Pythagorean Triple if and only if $(a/c, b/c)$ is a rational point on the unit circle.

Exercise 6. Find 6 rational points on the unit circle using Exercise 4, and then use each to find a nontrivial Pythagorean Triple.

How do we know that the analysis above leads to all possible Pythagorean triples? For any such triple (a, b, c) we know that $(a/c, b/c)$ is a rational point on the unit circle. But each rational point on the unit circle corresponds to many different triples. For example, (6, 8, 10) and (9, 12, 15) are both Pythagorean triples, but they both lead to the same rational point, (3/5, 4/5). Going the other direction, if we systematically produce all the rational points, how do we use those to generate *all* the possible Pythagorean triples? That is the question addressed by the next few definitions and exercises.

Definition 3. A Pythagorean triple (a, b, c) is called *primitive* if there is no common divisor of a, b, and c other than 1 or -1.

Exercise 7. Prove the following proposition: If p and q are relatively prime integers, one of which is even, then $(2pq, p^2 - q^2, p^2 + q^2)$ is a primitive Pythagorean triple.

Exercise 8. Prove the following proposition: If p and q are relatively prime odd integers, then $(pq, (p^2 - q^2)/2, (p^2 + q^2)/2)$ is a primitive Pythagorean triple.

Exercise 9. Prove the following proposition: Every primitive Pythagorean triple is either of the form $(pq, (p^2 - q^2)/2, (p^2 + q^2)/2)$ where p and q are relatively prime odd integers, or else it is of the form $(2pq, p^2 - q^2, p^2 + q^2)$ where p and q are relatively prime integers, one of which is even.

Exercise 2.20. Explain how the preceding three exercises, when combined with the work of Harvey, Hymernie, and Rong, can be used to generate ALL possible Pythagorean triples.