Day 22 Fri 4/6/2018

1. The general difference equation problem (assuming distinct roots of characteristic polynomial.
   a. We want to find an equation for the k^{th} term of a sequence defined by
   $$a_{k+n} = c_0 a_k + c_1 a_{k+1} + c_2 a_{k+2} \cdots + c_{n-1} a_{k+n-1} \qquad \text{(EQ 0)}$$
   with given values of $a_0, a_1, a_2, \cdots, a_{n-1}$.
   b. We can write the difference equation in the form
   $$L^n\{a_k\} = c_0\{a_k\} + c_1 L\{a_k\} + c_2 L^2\{a_k\} + \cdots + c_{n-1} L^{n-1}\{a_k\}$$
   or equivalently
   $$(L^n - c_0 - c_1 L - c_2 L^2 - \cdots - c_{n-1} L^{n-1})\{a_k\} = 0. \qquad \text{(EQ 1)}$$
   c. We recognize the characteristic polynomial as
   $p(t) = t^n - c_0 - c_1 t - c_2 t^2 - \cdots - c_{n-1} t^{n-1}$ and
   the companion matrix as $C = \left[\begin{array}{c|cccc} 0 & & & I & \\ \hline c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{array}\right]$
   d. For the results below, we assume that $p(t)$ has n distinct roots $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.
   e. As a preliminary step, we note that the solution to (EQ 1) with initial vector $(0, 0, 0, \cdots, 0, 1)$ has a known formula. We will refer to this solution as the sequence
   $$\{E_k\}_{k=0}^{\infty} = (E_0, E_1, E_2, E_3, \cdots) = (0, 0, \cdots, 0, 1, E_n, E_{n+1}, E_{n+2}, E_{n+3}, \cdots)$$
   and we know $E_k = \sum_{j=1}^{n} \frac{\lambda_j^k}{p'(\lambda_j)}$.
   f. In order to obtain our desired initial vector, we will form a linear combination of $\{E_k\}, L\{E_k\}, L^2\{E_k\}, \cdots, L^{n-1}\{E_k\}$. Specifically we set
   $$\{a_k\} = b_0\{E_k\} + b_1 L\{E_k\} + b_2 L^2\{E_k\} + \cdots + b_{n-1} L^{n-1}\{E_k\}$$
   and equate the first n terms of the sequences on either side of the equation to find the coefficients $b_0, \cdots, b_{n-1}$.
   g. We can do this efficiently by narrowing our focus to
   $$\{a_k\}_{k=0}^{n-1} = b_0\{E_k\}_{k=0}^{n-1} + b_1\{E_k\}_{k=1}^{n} + b_2\{E_k\}_{k=2}^{n+1} + \cdots + b_{n-1}\{E_k\}_{k=n-1}^{2n-2}$$
   and writing each of the finite sequences as column vectors. That produces the system
   $$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix} = b_0 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ E_n \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ E_n \\ E_{n+1} \end{bmatrix} + \cdots + b_{n-1} \begin{bmatrix} 1 \\ E_n \\ \vdots \\ E_{2n-4} \\ E_{2n-3} \\ E_{2n-2} \end{bmatrix}.$$
   h. This in turn can immediately be written as the matrix-vector equation

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & E_{2n-5} & E_{2n-4} \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix}.$$

i. As in the example this is recognized as a triangular system, and in any specific example we can solve it easily. Doing so in many examples and looking for patterns reveals a remarkable fact. The inverse of the matrix on the left is simply connected with the coefficients of the original difference equation. Specifically,

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & E_{2n-5} & E_{2n-4} \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix}^{-1} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & -c_4 & \cdots & 1 & 0 \\ -c_3 & -c_4 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{(EQ 2)}$$

2. A specific example

Reconsider the example $a_{k+4} = a_{k+3} + 2a_{k+2} - a_{k+1} - a_k$. This can be rewritten in the form $(L^4 - c_0 - c_1 L - c_2 L^2 - c_3 L^3)\{a_k\} = 0$ where $c_0 = -1,\ c_1 = -1,\ c_2 = 2,$ and $c_3 = 1$. We also know that the E sequence for this difference equation begins $(E_0, E_1, E_2, E_3, E_4, E_5, E_6, \cdots) = (0, 0, 0, 1, 1, 3, 4, \ \cdots)$. So for this example equation 2 specializes to

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$ We can verify this correct by multiplying

$$\begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This doesn't prove equation 2, but it may provide some confidence in it. Before proceeding to a proof of equation 2, let us apply it to find the solution of the general problem introduced in paragraph 8.

3. Returning to the equation in paragraph 8h, $$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & E_{2n-5} & E_{2n-4} \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix}$$

a. We can solve for the b vector by multiplying both sides by the inverse matrix given in equation 2. That gives $\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & -c_4 & \cdots & 1 & 0 \\ -c_3 & -c_4 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix}$. Since the difference equation coefficients (the c's) and the initial terms (a_0 through a_{n-1}) are givens, we may regard the *b's* as known. Therefore, from the equation in 8f,

$$\{a_k\} = b_0\{E_k\} + b_1 L\{E_k\} + b_2 L^2\{E_k\} + \cdots + b_{n-1}L^{n-1}\{E_k\},$$

we can extract an equation for any specific term a_k. Specifically

$$a_k = b_0 E_k + b_1 E_{k+1} + b_2 E_{k+2} + \cdots + b_{n-1}E_{k+n-1} = \sum_{m=0}^{n-1} b_m E_{k+m}.$$

b. But we know $E_k = \sum_{j=1}^{n} \frac{\lambda_j^k}{p'(\lambda_j)}$ and therefore $E_{k+m} = \sum_{j=1}^{n} \frac{\lambda_j^{k+m}}{p'(\lambda_j)} = \sum_{j=1}^{n} \lambda_j^m \frac{\lambda_j^k}{p'(\lambda_j)}$.

c. Substituting gives

$$a_k = \sum_{m=0}^{n-1} b_m \sum_{j=1}^{n} \lambda_j^m \frac{\lambda_j^k}{p'(\lambda_j)} = \sum_{m=0}^{n-1}\sum_{j=1}^{n} b_m \lambda_j^m \frac{\lambda_j^k}{p'(\lambda_j)}.$$

d. Next, interchange the two summations, and observe that the fraction does not depend on m. Thus

$$a_k = \sum_{j=1}^{n}\sum_{m=0}^{n-1} b_m \lambda_j^m \frac{\lambda_j^k}{p'(\lambda_j)} = \sum_{j=1}^{n} \left(\sum_{m=0}^{n-1} b_m \lambda_j^m\right)\frac{\lambda_j^k}{p'(\lambda_j)}$$

e. Now the inner summation is a polynomial in λ_j. In fact, if we define the polynomial $f(t) = \sum_{m=0}^{n-1} b_m t^m$, then the inner sum is $f(\lambda_j)$. Substituting in the prior equation gives the final result:

$$a_k = \sum_{j=1}^{n} f(\lambda_j)\frac{\lambda_j^k}{p'(\lambda_j)} = \sum_{j=1}^{n} \frac{f(\lambda_j)}{p'(\lambda_j)}\lambda_j^k.$$

4. Revisiting an earlier example

a. Reconsider the example from paragraph 7
$a_{k+4} = a_{k+3} + 2a_{k+2} - a_{k+1} - a_k$; $a_0 = 10,\ a_1 = 13,\ a_2 = 17,\ a_3 = 19.$

b. The difference equation has the form $a_{k+4} = c_3 a_{k+3} + c_2 a_{k+2} + c_1 a_{k+1} + c_0 a_k$ with $c_3 = 1$, $c_2 = 2$, $c_1 = -1$, and $c_0 = -1$.

c. The characteristic polynomial is $p(t) = t^4 - t^3 - 2t^2 + t + 1$.

d. We compute $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & 1 \\ -c_2 & -c_3 & 1 & 0 \\ -c_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \\ 17 \\ 19 \end{bmatrix} = \begin{bmatrix} -14 \\ -16 \\ 3 \\ 10 \end{bmatrix}$ and define $f(t) = 10t^3 + 3t^2 - 16t - 14$.

e. By our general result, $a_k = \sum_{j=1}^{4} \frac{f(\lambda_j)}{p'(\lambda_j)} \lambda_j^k$.

f. Now we have studied this difference equation before. We found the roots $\lambda_1, \lambda_2, \lambda_3$, and λ_4 to be 1, -1, $(1+\sqrt{5})/2$, and $(1-\sqrt{5})/2$. Substituting these into *p*'(*t*), also found $p'(\lambda_1) = p'(\lambda_2) = -2$, $p'(\lambda_3) = \sqrt{5}\,\lambda_3$, and $p'(\lambda_4) = -\sqrt{5}\,\lambda_4$.

g. Next we want to compute *f* of each root. By direct computation, $f(\lambda_1) = -17$ and $f(\lambda_2) = -5$. We know the remaining roots are also roots of the quadratic $g(t) = t^2 - t - 1$. Divide this into *f* (*t*) and find the remainder *r*(*t*): $10t^3 + 3t^2 - 16t - 14 = (t^2 - t - 1)(10t + 13) + (7t - 1)$ so $r(t) = 7t - 1$. Then $f(\lambda_3) = r(\lambda_3) = (5 + 7\sqrt{5})/2$ and $f(\lambda_4) = r(\lambda_4) = (5 - 7\sqrt{5})/2$.

h. Combining all of these results, we have the equation

$$a_k = \frac{17}{2} + \frac{5}{2}(-1)^k + \frac{5 + 7\sqrt{5}}{\sqrt{5}(1+\sqrt{5})}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{(5 - 7\sqrt{5})}{\sqrt{5}(1-\sqrt{5})}\left(\frac{1-\sqrt{5}}{2}\right)^k .$$

This simplifies to

$$a_k = \frac{17}{2} + \frac{5}{2}(-1)^k + \frac{-1 + 3\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1 + 3\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^k .$$

5. To complete this discussion, we take up the proof of equation 2. To do so, we will verify that

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & E_{2n-5} & E_{2n-4} \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix} \cdot \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & -c_4 & \cdots & 1 & 0 \\ -c_3 & -c_4 & -c_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = I \qquad \text{(EQ 3)}$$

a. As a first step, let us represent a general row of the matrix on the left. Notice that there are *n* – 1 zeros in the first row, *n* – 2 zeros in the second row, and in general, *n* – *i* zeros in row *i*. This shows that row *i* of the left matrix has the form

$$[\underbrace{0 \quad 0 \quad 0 \quad \cdots \quad 0}_{n-i} \quad 1 \quad \underbrace{E_n \quad E_{n+1} \quad \cdots \quad E_*}_{i-1}] .$$

Notice the subscript on the final *E* is left unspecified. It is easy to see that it has to be

$n + i - 2$, but that fact will not be needed, and including it is an unnecessary complication.

b. In a similar way, we can represent a general column of the second matrix. However, let us write that column as a row, to save space and to simplify computing dot products. Now observe that the first entry in column j is $-c_j$. Also, there are no zeros in column 1, 1 zero in column 2, two zeros in column 3, and in general, $j - 1$ zeros in column j. That means, written horizontally, column j takes the form

$$[\underbrace{-c_j \quad \cdots \quad -c_{n-3} \quad -c_{n-2} \quad -c_{n-1}}_{n-j} \quad 1 \quad \underbrace{0 \quad 0 \quad \cdots \quad 0}_{j-1}] \,.$$

c. Now we can compute the dot product of row i of the first matrix with column j of the second. We consider three cases. First, suppose that $i = j$. In this case the 1 in row i aligns with the 1 in column j, because each 1 is followed by $i - 1 = j - 1$ entries. Every other position has either a zero in row i or a zero in column j. So when $i = j$, row i dotted with column j produces 1.

d. For case 2, suppose that $i < j$. Then there are more 0's to the right of the 1 in column j than there are entries to the right of 1 in row i, so the 1 in column j is to the *left* of the 1 in row i. Displaying both row i and column j together produces this result:

$$\begin{array}{ll} \text{row } i: & [\;0 \quad \cdots \quad 0 \quad 0 \quad 0 \;\cdots\; 0 \quad 1 \quad E_n \quad \cdots \quad E_*] \\ \text{column } j: & [-c_j \quad \cdots \quad -c_{n-1} \quad 1 \quad 0 \;\cdots\; 0 \quad 0 \quad 0 \quad \cdots \quad 0\,] \,. \end{array}$$

Here it is evident that the dot product is 0.

e. The final case is $i > j$. This time the 1 in column j is to the *right* of the 1 in row i, leading to the following diagram:

$$\begin{array}{ll} \text{row } i: & [\;0 \quad \cdots \quad 0 \quad 1 \quad E_n \quad \cdots \quad E_* \quad E_* \quad E_* \quad \cdots \quad E_*] \\ \text{column } j: & \left[-c_j \quad \cdots \quad -c_* \quad -c_* \quad -c_* \quad \cdots \quad -c_{n-1} \quad 1 \quad \underbrace{0 \quad \cdots \quad 0}_{j-1}\right] . \end{array}$$

As before, the subscripts that appear as *'s are left unspecified.

Notice that the final $j - 1$ entries of column j are all 0. Therefore, before computing the dot product we can eliminate these entries of each vector, resulting in

$$\begin{array}{ll} \text{row } i: & [\;0 \quad \cdots \quad 0 \quad 1 \quad E_n \quad \cdots \quad E_* \quad E_*\,] \\ \text{column } j: & \left[-c_j \quad \cdots \quad -c_* \quad -c_* \quad -c_* \quad \cdots \quad -c_{n-1} \quad 1\,\right] . \end{array}$$

The dot product of these vectors is the same as the dot product of the original two vectors.

Next, let us add j entries to the front of each vector, inserting zeros in the top vector, and $-c_0, -c_1, \cdots, -c_{j-1}$ in the bottom vector. That produces

$$\begin{array}{ll} \text{row } i: & [\;0 \quad 0 \quad \cdots \quad 0 \quad 0 \;\cdots\; 0 \quad 1 \quad E_n \;\cdots\; E_* \quad E_*\,] \\ \text{column } j: & \left[-c_0 \quad -c_1 \quad \cdots \quad -c_{j-1} \quad -c_j \;\cdots\; -c_* \quad -c_* \quad -c_* \;\cdots\; -c_{n-1} \quad 1\,\right] . \end{array}$$

Again, this change has not altered the value of the dot product.

Now we will argue that the dot product of these two vectors is zero. We make several observations. First, each vector has exactly $n + 1$ entries, because we eliminated $j - 1$ entries from the right sides and inserted j entries on the left. Second, there can be at most $n - 1$ zeros at the start of the top vector. In fact, we know that there were originally $n - i$ zeros, and then we inserted another j zeros, so there must be exactly $n - i + j = n - (i - j)$ zeros. But we know that $i > j$ so $i - j$ is at least 1, and that implies that the number of zeros is at most $n - 1$. Third, we can now recognize that the entries of the top vector are $n + 1$ *consecutive* terms of the E sequence. This is true because the 1 is equal to E_{n-1}, and all of the preceding
$n - 1$ E terms (from E_0 to E_{n-2}) equal 0. Since there are at most $n - 1$ zeros in the vector, we can replace them with consecutive E terms ending with E_{n-2}. Let the first of these E terms be E_r, where $r \geq 0$. Then the top vector is made up of all the terms from E_r to E_{r+n}, and our diagram can be expressed as

$$\begin{array}{ll} \text{row } i: & [\; E_r \quad E_{r+1} \quad E_{r+2} \quad \cdots \quad E_{r+n-1} \quad E_{r+n} \;] \\ \text{column } j: & [-c_0 \quad -c_1 \quad -c_2 \quad \cdots \quad -c_{n-1} \quad 1 \;]\;. \end{array}$$

This dot product equals zero, because we know that the E sequence obeys our given difference equation (equation 0 at the start of the outline), and hence that

$$E_{r+n} = c_0 E_r + c_1 E_{r+1} + c_2 E_{r+2} \cdots + c_{n-1} E_{r+n-1}.$$

Thus, we have shown that in case 3, when $i > j$, the dot product of row i with column j equals 0.

To summarize, when we multiply the two matrices in equation (3), row i of the first matrix times column j of the second matrix produces 1 when $i = j$ and otherwise produces 0. That proves the matrix product is the identity matrix, which is what we wanted to show.

End of Day

2nd Exercise Set on Lag Operator and Difference Equations

In problems 1 through 4, a difference equation and set of initial terms is given. In completing each problem, let $\{E_k\}$ be the sequence that is defined by the given difference equation when the initial terms are 0, 0, ... , 1 (as opposed to the initial terms given in the problem). Note that except for problem 4, these are the sequences you analyzed in problem 1 of the preceding problem set. For each problem you are to complete the following steps:

a. Find the matrix $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix}$ from equation 2 of the lecture outline. Use the difference equation to find the necessary values of E_n, E_{n+1}, etc..

b. Find the matrix $\begin{bmatrix} -c_1 & -c_2 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$ from equation 2 of the lecture outline.

c. Multiply the matrices from steps i and ii together and verify that they are inverses of each other.

d. Compute the coefficient vector $\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & -c_4 & \cdots & 1 & 0 \\ -c_3 & -c_4 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix}$ for the given difference equation and initial terms.

e. Verify that $a_k = b_0 E_k + b_1 E_{k+1} + b_2 E_{k+2} + \cdots + b_{n-1} E_{k+n-1}$ holds for $k = 0, 1, ..., n - 1$.

f. Explain why these steps imply that the sequence defined by

$$\{a_k\} = b_0\{E_k\} + b_1 L\{E_k\} + b_2 L^2\{E_k\} + \cdots + b_{n-1} L^{n-1}\{E_k\}$$

is the solution to the given difference equation with the specified initial terms.

g. Find the polynomial $f(t) = \sum_{m=0}^{n-1} b_m t^m$ discussed in the lecture outline paragraph 3e.

h. Express a_k as a function of k in the form $\sum_{j=1}^{n} \frac{f(\lambda_j)}{p'(\lambda_j)} \lambda_j^k$ where $p(t)$ is the characteristic polynomial for the given difference equation and $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, are the roots of $p(t)$.

1. $a_{k+2} = 2a_{k+1} + a_k$; $a_0 = 2,\ a_1 = 3$

2. (★) $a_{k+3} = 3a_{k+2} - 2a_k$; $a_0 = 3,\ a_1 = 1,\ a_2 = 5$
   Comment: compare your answer for part e with your answer to problem 5 of the previous problem set.

3. $a_{k+4} = 4a_{k+3} - 6a_{k+2} + 4a_{k+1} + 8a_k$; $a_0 = 2,\ a_1 = 0,\ a_2 = -3,\ a_3 = 2$
   Hints: Observe that $p(t) = (t-1)^4 - 9$. This allows you to find the roots as $1 \pm \sqrt{3}$ and $1 \pm i\sqrt{3}$. Also, $p'(t) = 4(t-1)^3$. This is easy to compute when t is any of the roots of p. For example, $p'(1+\sqrt{3}) = 4(\sqrt{3})^3 = 12\sqrt{3}$.

   Answer for part h: $$\frac{(5-\sqrt{3})(1-\sqrt{3})^k + (5+\sqrt{3})(1+\sqrt{3})^k + (7-5i\sqrt{3})(1-i\sqrt{3})^k + (7+5i\sqrt{3})(1+i\sqrt{3})^k}{12}$$

4. $a_{k+3} = a_{k+2} + 8a_{k+1} + 6a_k$; $a_0 = 1,\ a_1 = -4,\ a_2 = 6$
   Hint: The characteristic polynomial has a root you can find by guessing.

5. $a_{k+5} = 2\,a_{k+4} + 3a_{k+3} + 3a_{k+2} + 2a_{k+1} - a_k$; $a_0 = a_1 = a_2 = a_3 = a_4 = 1.$
   Comment: The characteristic polynomial for this problem is a palindromial, and its roots can be found using methods we have studied before. However, they are too complicated to substitute into either $p'(t)$ or $f(t)$. So for this problem, you are not required to complete part h.

Solution for problem 2

a. Here $n = 3$, so we are to find the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & E_3 \\ 1 & E_3 & E_4 \end{bmatrix}$. We find $E_3 = 3E_2 - 2E_0 = 3$ and $E_4 = 3E_3 - 2E_1 = 9$. Thus, the matrix we want is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 9 \end{bmatrix}$.

b. Again using $n = 3$, we seek the matrix $\begin{bmatrix} -c_1 & -c_2 & 1 \\ -c_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. From the difference equation $a_{k+3} = 3a_{k+2} - 2a_k = c_2a_{k+2} + c_1a_{k+1} + c_0a_k$ we see that $c_1 = 0$ and $c_2 = 3$. Therefore the desired matrix is $\begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

c. Direct computation shows that $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

d. By direct computation $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 3 \end{bmatrix}$

e. We want to verify that $a_k = 2E_k - 8E_{k+1} + 3E_{k+2}$ holds for $k = 0, 1, 2$. With $k = 0$, we ask whether $3 = 2E_0 - 8E_1 + 3E_2$. Since $(E_0,\ E_1,\ E_2) = (0, 0, 1)$, the desired equation clearly holds. The other two equations are verified similarl, using the facts that $(a_0,\ a_1,\ a_2) = (3, 1, 5)$, $(E_1,\ E_2,\ E_3) = (0, 1, 3)$, and $(E_2,\ E_3,\ E_4) = (1, 3, 9)$.

f. First, note that with $n = 3$ and $(b_0,\ b_1,\ b_2) = (2, -8, 3)$, the equation in the problem statement becomes $\{a_k\} = 2\{E_k\} - 8L\{E_k\} + 3L^2\{E_k\}$. We know that $L\{E_k\}$ and $L^2\{E_k\}$ are both solutions of the difference equation, so any linear combination of them is also a solution. This shows that the right side of the equation is a solution of the difference equation. Next note that by part e, the initial terms of the right hand side of the equation are respectively equal to $a_0,\ a_1,$ and a_2. Therefore, the right hand side is the solution to the given difference equation with the specified initial terms.

g. The polynomial $f(t) = \sum_{m=0}^{n-1} b_m t^m = 2 - 8t + 3t^2$.

h. The characteristic polynomial for this difference equation is $p(t) = t^3 - 3t^2 + 2 = (t-1)(t^2 - 2t - 2)$. The roots are 1 and $1 \pm \sqrt{3}$. We differentiate the characteristic polynomial to find $p'(t) = 3t^2 - 6t = 3t(t-2)$. This leads to $p'(1) = -3$, $p'(1+\sqrt{3}) = 3(1+\sqrt{3})(-1+\sqrt{3}) = 6$, and $p'(1-\sqrt{3}) = 3(1-\sqrt{3})(-1-\sqrt{3}) = 6$. We also calculate $f(1) = -3$ and $f(1 \pm \sqrt{3}) = 2 - 8(1 \pm \sqrt{3}) + 3(4 \pm 2\sqrt{3}) = 6 \mp 2\sqrt{3}$. Substituting these into the expression $\sum_{j=1}^{3} \frac{f(\lambda_j)}{p'(\lambda_j)} \lambda_j^k$ produces

$$a_k = 1 + \frac{3-\sqrt{3}}{3}\left(1+\sqrt{3}\right)^k + \frac{3+\sqrt{3}}{3}\left(1-\sqrt{3}\right)^k.$$