Day 4 Fri 1/26/2018

Overview: matrices and polynomials

1. given a polynomials $p(t)$ and a square matrix A we can form $p(A)$
2. We can make a polynomial whose coefficients are matrices
3. We can make a matrix whose entries are polynomials
4. Note that 2 and 3 are really the same thing
5. We can associate a polynomial with a matrix. That is, we can define a mapping from the set of matrices into the set of polynomials.
   a. The minimal polynomial for a square matrix A is a monic polynomial $m(x)$ of minimal degree for which $m(A)$ = zero matrix
   b. The characteristic polynomial for a square matrix A (n x n) is a monic polynomial $p_A(t)$ of degree n with the property that $p_A(\alpha) = 0$ iff α is an eigenvalue. Defined by $p_A(t) = \det(tI - A)$
   c. Eigenvalues are very important for most of the course, and we will review them next time.
6. We can associate a matrix with a polynomial. That is, we can define a mapping from the set of polynomials into the set of matrices. For example, the companion matrix of a the monic polynomial
$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$
is the n x n matrix as shown here:
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$
The companion matrix of p has p as its characteristic polynomial. So this association is the inverse of the one in 5b.

Matrix patterns: powers of A when A has some pattern of entries; $p(A)$ for polynomial p

1. diagonal matrices
2. N and N^T ; relating pattern to row operation matrix concept; $I + N$
3. Companion matrix for a polynomial and recursively defined sequence. Look at $M^k\mathbf{v}$ for a fixed numerical vector $\mathbf{v}$. This can be used to get a formulation for M^k itself, but we won't do that now.
4. $M = N + N^T$
   a. powers of M for 3x3 case
      i. Direct computation: $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $M^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
      ii. freemat to generate patterns of entries
      iii. notice all entries are powers of 2

iv. pattern appears to show that
$M^k = 2^{(k-1)/2}M$ if k is odd and $M^k = 2^{(k-2)/2}M^2$ if k is even
v. Prove by induction?

b. Using the pattern to evaluate $p(M)$
i. Use the formulas to change x into a combination of M and M^2
ii. Observe that $M^3 - 2M = 0$. This can be used to simplify evaluation as follows. Divide $p(t)$ by $t^3 - 2t$, and find the remainder $r(t)$. Then $p(M) = r(M)$. Proof: by definition of remainder, we have $p(t) = q(t)(t^3 - 2t) + r(t)$. Substitute M in place of t, and use the fact that $M^3 - 2M = 0$.
iii. The computations are really equivalent in the two approaches above, but the second approach is more generally applicable, even when there is no evident pattern in the powers of a matrix.

c. 4 x 4 case
i. freemat to generate patterns of entries
ii. notice all entries are fib numbers
iii. pattern appears to show:

$$M^k = \begin{bmatrix} 0 & F_k & 0 & F_{k-1} \\ F_k & 0 & F_{k+1} & 0 \\ 0 & F_{k+1} & 0 & F_k \\ F_{k-1} & 0 & F_k & 0 \end{bmatrix}$$ for odd k and

$$M^k = \begin{bmatrix} F_{k-1} & 0 & F_k & 0 \\ 0 & F_{k+1} & 0 & F_k \\ F_k & 0 & F_{k+1} & 0 \\ 0 & F_k & 0 & F_{k-1} \end{bmatrix}$$ for even k

d. Concept of minimal polynomial and repeat of how to reduce $p(x)$ mod the min polynomial and then evaluate $p(M)$

End of Day