

Overview: matrices and polynomials

1. given a polynomials $p(t)$ and a square matrix A we can form $p(A)$
2. We can make a polynomial whose coefficients are matrices
3. We can make a matrix whose entries are polynomials
4. Note that 2 and 3 are really the same thing
5. We can associate a polynomial with a matrix. That is, we can define a mapping from the set of matrices into the set of polynomials.
 - a. The minimal polynomial for a square matrix A is a monic polynomial $m(x)$ of minimal degree for which $m(A) = \text{zero matrix}$
 - b. The characteristic polynomial for a square matrix A ($n \times n$) is a monic polynomial $p_A(t)$ of degree n with the property that $p_A(\alpha) = 0$ iff α is an eigenvalue. Defined by $p_A(t) = \det(tI - A)$
 - c. Eigenvalues are very important for most of the course, and we will review them next time.
6. We can associate a matrix with a polynomial. That is, we can define a mapping from the set of polynomials into the set of matrices. For example, the companion matrix of a the monic polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

is the $n \times n$ matrix as shown here:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

The companion matrix of p has p as its characteristic polynomial. So this association is the inverse of the one in 5b.

Matrix patterns: powers of A when A has some pattern of entries; $p(A)$ for polynomial p

1. diagonal matrices
2. N and N^T ; relating pattern to row operation matrix concept; $I + N$
3. Companion matrix for a polynomial and recursively defined sequence. Look at $M^k \mathbf{v}$ for a fixed numerical vector \mathbf{v} . This can be used to get a formulation for M^k itself, but we won't do that now.
4. $M = N + N^T$
 - a. powers of M for 3x3 case

- i. Direct computation: $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $M^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

- ii. freemat to generate patterns of entries
 - iii. notice all entries are powers of 2

- iv. pattern appears to show that
 $M^k = 2^{(k-1)/2}M$ if k is odd and $M^k = 2^{(k-2)/2}M^2$ if k is even
- v. Prove by induction?
- b. Using the pattern to evaluate $p(M)$ for any polynomial p .
- Use the formulas to change M^k into a combination of M and M^2
 - Observe that $M^3 - 2M = 0$. This can be used to simplify evaluation as follows. Divide $p(t)$ by $t^3 - 2t$, and find the remainder $r(t)$. Then $p(M) = r(M)$. Proof: by definition of remainder, we have $p(t) = q(t)(t^3 - 2t) + r(t)$. Substitute M in place of t , and use the fact that $M^3 - 2M = 0$.
 - The computations are really equivalent in the two approaches above, but the second approach is more generally applicable, even when there is no evident pattern in the powers of a matrix.
 - Minimal Polynomial: We know that $M^3 - 2M = 0$, and in fact $M(M^2 - I) = M(M - I)(M + I) = 0$, but M is not equal to I or $-I$ or 0 , and M^2 is not equal to I . So M is not annihilated by any of the factors of $t^3 - 2t$. This means that $t^3 - 2t$ is the minimal polynomial for M . In general, if $m(t)$ is the minimal polynomial for A and $p(t)$ is any other polynomial that annihilates A , then $m(t)$ must be a factor of $p(t)$. Reason: Suppose not. Then by long division, we know we can write $p(t) = m(t)q(t) + r(t)$ where q is the quotient and r is the (nonzero) remainder, with degree less than the degree of $m(t)$. Since this is a polynomial identity, it must hold when we substitute A for t : $p(A) = m(A)q(A) + r(A)$. But both $p(A)$ and $m(A) = 0$, so we conclude that $r(A) = 0$. Thus $r(t)$ annihilates A but has lower degree than the minimal polynomial for A , which is impossible. This shows that the remainder r is actually 0, and that $m(t)$ is a factor of $p(t)$, as asserted.
- c. 4 x 4 case
- freemat to generate patterns of entries
 - notice all entries are fib numbers
 - pattern appears to show:
- $$M^k = \begin{bmatrix} 0 & F_k & 0 & F_{k-1} \\ F_k & 0 & F_{k+1} & 0 \\ 0 & F_{k+1} & 0 & F_k \\ F_{k-1} & 0 & F_k & 0 \end{bmatrix} \text{ for odd } k \text{ and}$$
- $$M^k = \begin{bmatrix} F_{k-1} & 0 & F_k & 0 \\ 0 & F_{k+1} & 0 & F_k \\ F_k & 0 & F_{k+1} & 0 \\ 0 & F_k & 0 & F_{k-1} \end{bmatrix} \text{ for even } k$$
- d. Repeat concept of minimal polynomial and how to reduce $p(x)$ mod the min polynomial and then evaluate $p(M)$

End of Day