Day 9 Tue 2/13/2018

Continue with outline from two prior days. This outline will probably take us at least three days.

5. Lots of matrices are not diagonalizable. In fact, we have this
   Theorem: a triangular (nondiagonal) matrix with all diagonal entries equal cannot be diagonalizable.

   Proof: Let A be an $n \times n$ triangular matrix with all diagonal entries equal to one value λ, and with at least one nonzero entry off the main diagonal. We know that the diagonal entries of A must be eigenvalues of A, and since these are all equal, we see that A has a single eigenvalue λ, with multiplicity n.

   Now suppose that A is diagonalizable. Then for some invertible matrix P we have $P^{-1}AP = D$, where D is the diagonal matrix all of whose entries equal λ. That is, $D = \lambda I$. But then we have $P^{-1}AP = \lambda I$ so $A = P\lambda IP^{-1} = \lambda PIP^{-1} = \lambda I$. This shows that A is a diagonal matrix, which is a contradiction. Therefore we conclude that A is not diagonalizable. ■

6. Special case: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable. This follows from the following
   Theorem: If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are *distinct* eigenvalues of A, with corresponding *nonzero* eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a linearly independent set. [Note: the word *distinct* in this statement means that no two of the eigenvalues are equal.]

   This is Theorem 2 in section 5.1 of the file eigenvs_Lay4E.pdf posted on blackboard, where you will find a complete proof. I will say a little about the main idea of the proof momentarily. But for now, let us see how this theorem leads to the preceding statement: *no repeated eigenvalues implies diagonalizability*. If A is $n \times n$, then the characteristic polynomial is degree n, so there will be n roots, aka eigenvalues. Assume they are all distinct. For each eigenvalue there must be a nonzero eigenvector, and now our theorem says that these n eigenvectors are linear independent. Putting them as columns into a matrix P, we know that P will be invertible, and hence A is diagonalizable.

7. Main idea of the proof of the independence result. Suppose A is matrix with 4 distinct eigenvalues, say 2, 3, -5, and 8. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be nonzero eigenvectors for these eigenvalues. And suppose that the eigenvectors are NOT linearly independent. That means one of the vectors is a combination of the others. Since the numbering purely a matter of notational convenience, we may as well assume that it is $\mathbf{v}_4$ that is dependent on the other vectors, so that we can write $\mathbf{v}_4 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some scalar c's. And the c's cannot all equal 0 because $\mathbf{v}_4 \neq \mathbf{0}$.

Now apply A to both sides of this equation, and we obtain
$A\mathbf{v}_4 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3$, and because the **v**'s are eigenvectors, that gives us
$8\mathbf{v}_4 = c_1 2\mathbf{v}_1 + c_2 3\mathbf{v}_2 - c_3 5\mathbf{v}_3$. But from our first equation we also know
$8\mathbf{v}_4 = c_1 8\mathbf{v}_1 + c_2 8\mathbf{v}_2 + c_3 8\mathbf{v}_3$. Therefore we can subtract to find
$6c_1\mathbf{v}_1 + 5c_2\mathbf{v}_2 + 13c_3\mathbf{v}_3 = \mathbf{0}$, and as already mentioned these c's cannot all equal 0. (and the coefficients in front of the c's are all nonzero, because they are differences of distinct eigenvalues). That means we can solve for one of the **v**'s as a linear combination of the other two.

To summarize, assuming there are four dependent nonzero eigenvectors with distinct eigenvalues leads to the conclusion that there are actually *three* dependent nonzero eigenvectors with distinct eigenvalues. But the argument that we used can actually work with any number of nonzero eigenvectors having distinct eigenvalues. So we can build a kind of reverse induction argument, showing that we can keep reducing the number of vectors until we have only one, and that gives us a contradiction.

8. For matrices with some repeated eigenvalues, in some cases diagonalization is possible and in others it is not. Our theorem above about triangular matrices with all equal diagonal elements provides many examples where diagonalization is impossible. On the other hand, for any diagonal matrix D and any invertible matrix P, we can define the matrix $A = PDP^{-1}$. This matrix is evidently diagonalizable, and by putting some repeated elements on the diagonal of D, we assure that A has some repeated eigenvalues.

9. Here is the most general result about diagonalizability. Let A be an $n \times n$ matrix, and let the distinct eigenvalues be $\lambda_1, \lambda_2, \cdots, \lambda_k$, with $k < n$. For each eigenvalue λ_j, the set of solutions to the equation $(A - \lambda_j I)\mathbf{v} = \mathbf{0}$ is a subspace of $\mathbb{C}^n$ or $\mathbb{R}^n$ (and is actually the null space of the matrix $A - \lambda_j I$). This is referred to as the eigenspace corresponding to λ_j. The dimension of the eigenspace is the maximum number of independent eigenvectors possible for λ_j. With this terminology, A is diagonalizable iff the dimensions of its eigenspaces sum to n. In this case, we can make a set of n independent eigenvectors by choosing as many independent eigenvectors as possible for each eigenvalue. Putting all of these eigenvectors as columns into a single matrix gives us an invertible matrix P that diagonalizes A.

New Topic: Circulant Matrices

1. Overview:
   a. We have seen circulants before: as polynomials in the cyclic permutation matrix W.
   b. These matrices have applications in areas such as difference equations and graph theory
   c. Our immediate goal will be to study their eigen-properties and diagonalization. These depend on the diagonalization of W.
   d. The eigenvalues of W are complex n^{th} roots of unity.

2. The matrix W
   a. The $n \times n$ version is obtained by shifting the top row of the x identity matrix to the bottom row. W has 1's on the first super diagonal, and a 1 in the n,1 position (bottom of column 1), and zeros everywhere else.
   b. The characteristic polynomial is $\det(\lambda I - W) = \lambda^n - 1$.
      Proof by induction. For $n = 2$, $\det(\lambda I - W) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1$. So assume the result holds for a particular $n \geq 2$, and consider the $n+1 \times n+1$ W. We expand $\det(\lambda I - W)$ by minors in the first column. The cofactor the λ in position 1,1 is an $n \times n$ upper triangular matrix with all diagonal entries equal to λ. So the first term of the cofactor expansion is $\lambda \cdot \lambda^n = \lambda^{n+1}$. The remaining nonzero entry in the first column is a -1 in the n+1, 1 position. The cofactor is an $n \times n$ lower triangular matrix with all diagonal entries equal to -1. Its determinant is $(-1)^n$. So this term of the cofactor expansion is $(-1)(-1)^{n+2}(-1)^n$ (this is (n+1, 1 entry)·(±sign for that position)·(det of the submatrix)). This simplifies to 1 because it is an even power of -1. Thus we have shown that $\det(\lambda I - W) = \lambda^{n+1} - 1$, completing the induction step, and the proof.
   c. Suppose that α is one of the n^{th} roots of unity, and so an eigenvalue of W. Then an eigenvector corresponding to α is $\mathbf{v}_\alpha = [1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1}]^T$. We can verify this by computing $W\mathbf{v}_\alpha$. Since we know that W is the row operation matrix for moving the first row down to the bottom, we see immediately that $W\mathbf{v}_\alpha = [\alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1} \quad 1]^T$. But we also know that $1 = \alpha^n$. Substituting that in the preceding equation gives
      $$W\mathbf{v}_\alpha = [\alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1} \quad \alpha^n]^T = \alpha[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1}]^T = \alpha\mathbf{v}_\alpha.$$
   d. To complete the diagonalization of W, we need to understand the nature of the n^{th} roots of unity.

3. The n^{th} roots of unity
   a. Any complex number $x + iy$ can be expressed in polar coordinates as $r(\cos\theta + i\sin\theta)$ where (r, θ) are the polar coordinates of the point (x,y).
   b. Suppose we multiply two complex numbers, $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$. By direct computation we find
      $$wz = rs[(\cos\phi\cos\theta - \sin\phi\sin\theta) + i(\cos\phi\sin\theta + \sin\phi\cos\theta)] .$$

Now using the angle sum formulas for sine and cosine, we recognize that

$$wz = rs[\cos(\theta+\phi) + i\sin(\theta+\phi)].$$

Conclusion: complex multiplication is equivalent to multiplying the radius values and adding the angle values of the polar coordinates forms of the factors.

c. Corollary: $[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta)$.

d. Corollary: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

e. Corollary: Let $\theta = 2\pi/n$. If $\omega = (\cos\theta + i\sin\theta)$, then $\omega^n = (\cos n\theta + i\sin n\theta) = (\cos 2\pi + i\sin 2\pi) = 1$. This ω is an n^{th} root of unity.

f. Corollary: For any integer k, $(\omega^k)^n = \omega^{kn} = (\omega^n)^k = 1^k = 1$. So every power of ω is an n^{th} root of unity.

g. Geometrically, the complex numbers ω, ω^2, ω^3, $\cdots$, ω^{n-1}, $\omega^n = 1$ are distinct points on the unit circle in the complex plane, spaced equally with polar coordinates angles of $0, \theta, 2\theta, \cdots, (n-1)\theta$. These are therefore n distinct roots of the polynomial $\lambda^n - 1$, and that implies that they are *all* the n^{th} roots of unity.

h. These points on the unit circle come in complex conjugate pairs. In fact, $\overline{\omega^k} = \omega^{n-k} = \frac{\omega^n}{\omega^k} = \frac{1}{\omega^k}$. That is, the complex conjugate of a power of ω is both the reflection across the x axis, and the reciprocal, of that power of ω.

i. Special case: If n is even, then the negative of any n^{th} root of unity is another n^{th} root of unity. That is because $(-\alpha)^n = (-1)^n(\alpha^n) = 1$ if n is even and α is an n^{th} root of unity. In the even n case, we can group the n^{th} roots of unity in quartets: $\alpha, \bar{\alpha}, -\alpha,$ and $-\bar{\alpha}$, where α is any n^{th} root of unity. These four points constitute a square inscribed in the unit circle, produced by reflecting any vertex across both axes and through the origin.

j. Examples: For small values of n, such as 3, 4, 5, 6, 8, and 10, we can express ω using square roots and integers without much difficulty. (Note that in general, $\lambda^n - 1 = (\lambda - 1)(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1)$ and the second factor is a palindromial.) Once we find ω, we can find all the remaining n^{th} roots of unity using conjugates and negatives. We will look at the cases for 3, 4, 5, and 6 in detail. The cases $n = 8$ and $n = 10$ are assigned as exercises.

k. Cube roots of unity: We already looked at this case. $\lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1)$. The eigenvalues are 1, ω, and $\bar{\omega}$, where $\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$.

l. Fourth roots of unity. We can approach this two ways. Using our standard formulation, we can take $\theta = 2\pi/4 = \pi/2$. Then $\omega = (\cos\pi/2 + i\sin\pi/2) = i$, and the other powers are $\omega^2 = -1$, and $\omega^3 = -i = \bar{\omega}$. We can reach the same conclusion by factoring $\lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$.

m. Sixth roots of unity. If ω is a cube root of unity, it is also a sixth root of unity, because $\omega^6 = (\omega^3)^2 = 1^2 = 1$. By similar reasoning, $-\omega$ is a cube root of -1, and hence a sixth root of unity. But none of these cube roots of -1 are cube roots of unity. Thus we have accounted for 6 sixth roots of unity: ± 1, $\pm\frac{1}{2}\left(-1 + i\sqrt{3}\right)$ and

$\pm\frac{1}{2}(-1 - i\sqrt{3})$. Geometrically, these are the three cube roots of unity equally spaced around the unit circle, and their diametrically opposite points.

n. Fifth roots of unity. We factor $\lambda^5 - 1 = (\lambda - 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)$ and the second factor is a palindromial. We have already found the roots of this second factor by identifying the roots as r, s, and their reciprocals, defining $u = r + 1/r$, $v = s + 1/s$, and setting $(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1) = (\lambda^2 + u\lambda + 1)(\lambda^2 + v\lambda + 1)$. We found the roots to be $\frac{1}{4}(-1+\sqrt{5}) \pm \frac{1}{4}i\left(\sqrt{10+2\sqrt{5}}\right)$ and $\frac{1}{4}(-1-\sqrt{5}) \pm \frac{1}{4}i\left(\sqrt{10-2\sqrt{5}}\right)$. We also know that one of these must be $\omega = (\cos 2\pi/5 + i \sin 2\pi/5)$, and because $2\pi/5 < \pi/2$, ω is in the first quadrant. This in turn says that the real and imaginary parts of ω are both positive. Inspecting the four nonreal roots listed above, we see that $\omega = \frac{1}{4}(-1+\sqrt{5}) + \frac{1}{4}i\left(\sqrt{10+2\sqrt{5}}\right)$. Similarly, $\omega^2 = (\cos 4\pi/5 + i \sin 4\pi/5)$, and because $\pi/2 < 4\pi/5 < \pi$, ω^2 is in the second quadrant. This shows that $\omega^2 = \frac{1}{4}(-1-\sqrt{5}) + \frac{1}{4}i\left(\sqrt{10-2\sqrt{5}}\right)$. By similar logic, $\omega^3 = \overline{\omega^2}$, and $\omega^4 = \overline{\omega}$.

o. An interesting result from Galois theory states that n^{th} roots of unity are always expressible as a combination of radical expressions. That is equivalent to saying that $\cos 2\pi/n$ and $\sin 2\pi/n$ are expressible in terms of radicals for all n. Quite an extensive table of these cosine and sine values are tabulated in Wikipedia. See https://en.wikipedia.org/wiki/Trigonometric_constants_expressed_in_real_radicals .

4. Diagonalizing the matrix W
   a. Fact: the eigenvalues are the n^{th} roots of unity
   b. Fact: W has n distinct eigenvalues
   c. Therefore: W is diagonalizable. The diagonal matrix has the powers of ω on the main diagonal; the corresponding P matrix has columns of the form $[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1}]^T$ where α is ω^k for some $k \in \{0, 1, 2, \cdots, n-1\}$.
   d. In fact, the P matrix is symmetric. The entries in column j are powers of ω^{j-1}; specifically, the entry in row i and column j is $(\omega^{j-1})^{i-1} = \omega^{(i-1)(j-1)}$. This is symmetric in i and j (interchanging i and j has no net effect). This shows that $P^T = P$.
   e. For the inverse of P, we use an interesting fact: $P\overline{P} = nI$. This implies $P^{-1} = \frac{1}{n}\overline{P}$.
   f. As mentioned earlier, for small values of n, such as 3, 4, 5, 6, 8, and 10, all the n^{th} roots of unity can be expressed using square roots and integers. This also gives us simple formulations of P and P^{-1} for these n. We will look at the cases for 3, 4, and 5, in detail. The cases $n = 6$ and $n = 8$ are assigned as exercises.

5. Diagonalizing a Circulant Matrix
   a. Recall that an $n \times n$ circulant matrix is expressible in the form $q(W)$ where q is a polynomial of degree $n - 1$ and W is the matrix obtained from the $n \times n$ identity matrix by moving the top row to the bottom.

b. We can relate the eigenvalues and eigenvectors of a circulant matrix to the eigenvalues and eigenvectors of W.

c. For example, if $W\mathbf{v} = \lambda\mathbf{v}$, then $q(W)\mathbf{v} = q(\lambda)\mathbf{v}$. This follows because $W^2\mathbf{v} = W(W\mathbf{v}) = W(\lambda\mathbf{v}) = \lambda W\mathbf{v} = \lambda^2\mathbf{v}$, and more generally, $W^k\mathbf{v} = \lambda^k\mathbf{v}$ for any positive integer k. Therefore, with $q(t) = \sum_{k=0}^{n-1} c_k t^k$, we have

$$q(W)\mathbf{v} = \sum_{k=0}^{n-1} c_k W^k \mathbf{v} = \sum_{k=0}^{n-1} c_k \lambda^k \mathbf{v} = \left(\sum_{k=0}^{n-1} c_k \lambda^k\right)\mathbf{v} = q(\lambda)\mathbf{v}.$$

d. Conclusion: the eigenvalues of $C = q(W)$ can be found by applying polynomial q to the n^{th} roots of unity; the corresponding eigenvectors are the unchanged eigenvectors of W.

e. We can reach the same conclusion by diagonalizing C as follows. We know W is diagonalizable: $P^{-1}WP = D$ where

   - D is a diagonal matrix
   - Its diagonal elements are powers of $\omega = (\cos 2\pi/n + i \sin 2\pi/n)$
   - P has k^{th} column $[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1}]^{\mathrm{T}}$ where α is ω^k
   - $P^{-1} = \frac{1}{n}\bar{P}$

   Then $W = PDP^{-1}$, so

$$\begin{aligned} C = q(PDP^{-1}) &= \textstyle\sum_{k=0}^{n-1} c_k (PDP^{-1})^k \\ &= \textstyle\sum_{k=0}^{n-1} c_k P D^k P^{-1} \\ &= P(\textstyle\sum_{k=0}^{n-1} c_k D^k)P^{-1} \\ &= Pq(D^k)P^{-1} \quad . \end{aligned}$$

   But this is a diagonalization of C because we know $q(D^k)$ is a diagonal matrix with diagonal entries $q(\omega^k)$. These are thus the eigenvalues of C and the columns of P are corresponding eigenvectors.

f. Note: although the matrix W has distinct eigenvalues, that need not be the case for C. We know that the eigenvalues of C are $q(1)$, $q(\omega)$, $q(\omega^2)$, $\cdots$, $q(\omega^{n-1})$, and these need not be distinct. For example, if $q(t) = t^2$ and $n = 4$, then $\omega = i$, $\omega^3 = -i$, and $q(\omega) = q(\omega^3) = -1$.

g. One way to compute the eigenvalues of C is using the fact that $P\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} q(1) \\ q(\omega) \\ \vdots \\ q(\omega^{n-1}) \end{bmatrix}$.

   This follows because P is equal to its own transpose. That means the k^{th} row of P is the transpose of the k^{th} column, which in turn consists of the powers of ω^k. Therefore, the k^{th} row of P equals $[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{n-1}]^{\mathrm{T}}$ where α is ω^k. Now multiply this by the column $\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$ and the result is evidently <u>$q(\alpha)$.</u>

6. Example: Let $C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$.

   a. $C = q(W)$ where W is the 4 × 4 identity with top row moved to the bottom and with $q(t) = 1 + 2t + t^2 + 3t^3$. The coefficient vector is **c** = $[1 \quad 2 \quad 1 \quad 3]^T$.

   b. We already know that the fourth roots of unity are 1, i, -1, and $-i$.

   c. Taking powers of the eigenvalues gives us $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$. We also know that $P^{-1} = \frac{1}{4}\bar{P} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$

   d. The eigenvalues of C can be computed using P**c.** That gives
   $$\begin{bmatrix} q(1) \\ q(i) \\ q(-1) \\ q(-i) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -i \\ -3 \\ i \end{bmatrix}.$$

   e. The diagonalization equations are $P^{-1}CP = D$ and $C = PCP^{-1}$, where D is the diagonal matrix with diagonal entries of 7, $-i$, -3, and i.

   f. Knowing the eigenvalues, we can work out the characteristic polynomial for C:
   $$\begin{aligned} p(t) &= (t-7)(t+3)(t-i)(t+i) \\ &= (t^2-4t-21)(t^2+1) \\ &= t^4-4t^3-20t^2-4t-21. \end{aligned}$$
   Here I emphasize an important idea: Starting from a given circulant matrix, we computed *BOTH* the eigenvalues and the characteristic polynomial. This gives us the quartic polynomial $t^4-4t^3-20t^2-4t-21$ as well as its roots.

7. Example: 5 × 5 Circulant C at right
   $$\begin{bmatrix} 2 & 1 & 3 & 1 & -1 \\ -1 & 2 & 1 & 3 & 1 \\ 1 & -1 & 2 & 1 & 3 \\ 3 & 1 & -1 & 2 & 1 \\ 1 & 3 & 1 & -1 & 2 \end{bmatrix}$$

   a. $C = q(W)$ where W is 5 × 5 identity with top row moved to the bottom and with $q(t) = 2 + t + 3t^2 + t^3 - t^4$.

   b. We already know that the fifth roots of unity are 1, $\omega = \frac{1}{4}(-1+\sqrt{5}) + \frac{1}{4}i\left(\sqrt{10+2\sqrt{5}}\right)$, $\omega^2 = \frac{1}{4}(-1-\sqrt{5}) + \frac{1}{4}i\left(\sqrt{10-2\sqrt{5}}\right)$, $\omega^3 = \overline{\omega^2}$, and $\omega^4 = \bar{\omega}$.

   c. The eigenvalues of C are $q(1) = 6,\ q(\omega),\ q(\omega^2),\ q(\bar{\omega}) = \overline{q(\omega)},\ and\ q\left(\overline{\omega^2}\right) = \overline{q(\omega^2)}$.

   d. To compute $q(\omega)$ and $q(\omega^2)$, we use the fact that ω and ω^2 are both roots of $f(t) = 1 + t + t^2 + t^3 + t^4$. Thus we can divide $q(t)$ by $f(t)$ and then substitute ω and ω^2 into the remainder. Moverover, since f and q are of equal degree, the remainder will be found after a single step of division. Thus, we observe that $q(t) = f(t) + [q(t) - f(t)]$ showing that the remainder is $q(t) - f(t) = 1 + 2t^2 - 2t^4 = r(t)$. Substituting ω gives $q(\omega) = f(\omega) + r(\omega) = r(\omega)$, and similarly we derive $q(\omega^2) = r(\omega^2)$.

e. Continuing:
$$q(\omega) = r(\omega) = 1 + 2\omega^2 - 2\omega^4 = 1 + 2\omega^2 - 2\bar{\omega}$$
and
$$q(\omega^2) = r(\omega^2) = 1 + 2\omega^4 - 2\omega^8 = 1 + 2\bar{\omega} - 2\overline{\omega^2}.$$

f. Using our known value of ω, substitution and simplification lead to
$$q(\omega) = 1 - \sqrt{5} + \frac{i}{2}\left(\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}\right).$$
The quantity in parentheses simplifies, as follows. Let $\alpha = \sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}$. Then $\alpha > 0$ and
$$\begin{aligned}\alpha^2 &= \left(10 + 2\sqrt{5}\right) + \left(10 - 2\sqrt{5}\right) + 2\sqrt{10 + 2\sqrt{5}} \cdot \sqrt{10 - 2\sqrt{5}} \\ &= 20 + 2\sqrt{100 - 20} = 20 + 2\sqrt{80} = 20 + 8\sqrt{5} = 4\left(5 + 2\sqrt{5}\right).\end{aligned}$$
Thus
$$\alpha = 2\sqrt{5 + 2\sqrt{5}}.$$
Substituting this into our equation for $q(\omega)$ produces the eigenvalue
$$q(\omega) = 1 - \sqrt{5} + i\sqrt{5 + 2\sqrt{5}}.$$

g. The computation of $q(\omega^2)$ is similar, and gives us another eigenvalue. Together with their complex conjugates, this gives all four non-real eigenvalues of C, expressed compactly as
$$1 - \sqrt{5} \pm i\sqrt{5 + 2\sqrt{5}} \quad \text{and} \quad 1 + \sqrt{5} \pm i\sqrt{5 - 2\sqrt{5}} \ ,$$
and of course we have the real eigenvalue 6.

h. Eigenvectors. We know that for each eigenvalue λ of W an eigenvector is given by $\mathbf{v}_\lambda = \begin{bmatrix}1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4\end{bmatrix}^{\boldsymbol{T}}$ and that the P matrix is Vandermonde with columns of the form $\mathbf{v}_\lambda$. Every eigenvalue λ is a power of ω, so its powers are likewise powers of ω, and therefore are themselves eigenvalues. That means every entry of P must be $1, \omega, \bar{\omega}, \omega^2$, or $\overline{\omega^2}$. At the same time, powers of ω can be reduced modulo 5 because $\omega^5 = 1$. This leads to
$$P = \begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \overline{\omega^2} & \bar{\omega} \\ 1 & \omega^2 & \bar{\omega} & \omega & \overline{\omega^2} \\ 1 & \overline{\omega^2} & \omega & \bar{\omega} & \omega^2 \\ 1 & \bar{\omega} & \overline{\omega^2} & \omega^2 & \omega\end{bmatrix}.$$

We observe, as expected, that the P matrix is symmetric. We also know that $P\bar{P} = 5I$, so
$$P^{-1} = \frac{1}{5}\bar{P} = \frac{1}{5}\begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & \bar{\omega} & \overline{\omega^2} & \omega^2 & \omega \\ 1 & \overline{\omega^2} & \omega & \bar{\omega} & \omega^2 \\ 1 & \omega^2 & \bar{\omega} & \omega & \overline{\omega^2} \\ 1 & \omega & \omega^2 & \overline{\omega^2} & \bar{\omega}\end{bmatrix}.$$

i. Characteristic Polynomial. We can compute the characteristic polynomial from the factored form using the known values of the eigenvalues:
$$p(t) = (t-6)\left(t - 1 + \sqrt{5} + i\sqrt{5+2\sqrt{5}}\right)\left(t - 1 + \sqrt{5} - i\sqrt{5+2\sqrt{5}}\right)\left(t - 1 - \sqrt{5} + i\sqrt{5-2\sqrt{5}}\right)\left(t - 1 - \sqrt{5} - i\sqrt{5-2\sqrt{5}}\right)$$
In general, for a complex number $c = a + bi$, the product $(t-c)(t-\bar{c}\,)$ simplifies to
$$(t - a - bi)(t - a + bi\,) = (t-a)^2 + b^2 = t^2 - 2at + a^2 + b^2 \ .$$

Applying this to the two pairs of conjugate factors in the equation for $p(t)$ shows that

$$\begin{aligned}
p(t) &= (t-6)\left(t^2 - 2(1-\sqrt{5})t + (1-\sqrt{5})^2 + (5+2\sqrt{5})\right)\left(t^2 - 2(1+\sqrt{5})t + (1+\sqrt{5})^2 + (5-2\sqrt{5})\right) \\
&= (t-6)\left(t^2 - 2(1-\sqrt{5})t + (6-2\sqrt{5}) + (5+2\sqrt{5})\right)\left(t^2 - 2(1+\sqrt{5})t + (6+2\sqrt{5}) + (5-2\sqrt{5})\right) \\
&= (t-6)\left(t^2 - 2(1-\sqrt{5})t + 11\right)\left(t^2 - 2(1+\sqrt{5})t + 11\right) \\
&= (t-6)\left(t^2 - 2t + 11 + 2\sqrt{5}t\right)\left(t^2 - 2t + 11 - 2\sqrt{5}t\right) \\
&= (t-6)[(t^2 - 2t + 11)^2 - 20t^2] \\
&= (t-6)[t^4 - 4t^3 + 26t^2 - 44t + 121 - 20t^2] \\
&= (t-6)(t^4 - 4t^3 + 6t^2 - 44t + 121) \\
&= t^5 - 10t^4 + 30t^3 - 80t^2 + 375t - 726.
\end{aligned}$$

This illustrates again that, given a specific circulant matrix, we can find BOTH the eigenvalues and the characteristic polynomial. In this case, we have derived a specific quintic polynomial for which we know the roots, but for which we could probably not have found the roots otherwise. **[Actually, we *might* have been able to find them by guessing the integer root (6), which is a small factor of 726, and discovering that the quartic factor can be changed into a palindromial by making the substitution $t = \sqrt{11}z$.]**

End of Lecture

Exercises on Diagonalizability, Roots of Unity, and Circulant Matrices

1. (★) Let A be the 6×6 triangular matrix with all 3's on the main diagonal, a 1 in the 1,6 position, and 0's elsewhere. Is A diagonalizable?

2. (★) Let A be the 6×6 triangular matrix with diagonal entries 3, 3, 3, 5, 5, 5, a 1 in the 1,6 position, and 0's elsewhere. Find all eigenvalues and the corresponding eigenspaces. Is A diagonalizable?

3. Let A be the 6×6 triangular matrix with diagonal entries 3, 3, 3, 5, 5, 5, a 1 in the 1,2 position, and 0's elsewhere. Is A diagonalizable? Find all eigenvalues and the corresponding eigenspaces. Is A diagonalizable?

4. Find exact radical expressions for all of the (★)8^{th} and 10^{th} roots of unity. [Hint: Use the same method as in the example of 3m of the outline.]

5. For $n = 3$, (★)6, and 8, find the P matrix for the $n \times n$ W and also find P^{-1}.

6. (★) Let $C = \begin{bmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{4} \\ \sqrt[3]{4} & 1 & \sqrt[3]{2} \\ \sqrt[3]{2} & \sqrt[3]{4} & 1 \end{bmatrix}$. Find the eigenvalues and characteristic polynomial of C.

7. Let $C = \begin{bmatrix} 3 & -2 & 5 & 7 \\ 7 & 3 & -2 & 5 \\ 5 & 7 & 3 & -2 \\ -2 & 5 & 7 & 3 \end{bmatrix}$. Find a diagonalization of C. That means to find an invertible matrix P and a diagonal matrix D for which $P^{-1}CP = D$ and $C = PCP^{-1}$.

8. [OPTIONAL] Find all the 20^{th} roots of unity as exact radical expressions. You do not have to list all 20 individually, but can find a few (such as 1, ω, ω^2, and ω^3) and represent the others as combinations such as $-\omega, \overline{\omega},$ and so forth. [Hint: generalize the method used to find 6^{th} roots of unity in example 3m of the outline. Check your answers against the information shown in Wikipedia. See https://en.wikipedia.org/wiki/Trigonometric_constants_expressed_in_real_radicals .

9. [OPTIONAL]: Suppose that a matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. For each j let S_j be a linearly independent set of (one or more) eigenvectors for λ_j. Prove that the union $S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

Selected Solutions: Diagonalizability, Roots of Unity, and Circulant Matrices

1. (★) Let A be the 6×6 triangular matrix with all 3's on the main diagonal, a 1 in the 1,6 position, and 0's elsewhere. Is A diagonalizable?

   Matrix A is not diagonalizable. This is an application of the theorem stated in the first outline item on page 21.

2. (★) Let A be the 6×6 triangular matrix with diagonal entries 3, 3, 3, 5, 5, 5, a 1 in the 1,6 position, and 0's elsewhere. Find all eigenvalues and the corresponding eigenspaces. Is A diagonalizable?

   Matrix A is diagonalizable. Notice that there are two eigenvalues, 3 and 5. Two find eigenvectors we row reduce $A - 3I$ and $A - 5I$. But

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad A - 5I = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   and we can see that in each case the rref will have three pivots. That implies three free variables so we the eigenspace for each eigenvalue has dimension 3. By the theorem in the last item on page 22, A is diagonalizable. The eigenvectors for 3 have the form $[x \ \ y \ \ z \ \ 0 \ \ 0 \ \ 0\,]^T$ with x, y, and z free. The eigenvectors for 5 have the form $[z/2 \ \ 0 \ \ 0 \ \ x \ \ y \ \ z\,]^T$ with x, y, and z free.

4. Find exact radical expressions for all of the (★)8th roots of unity.

   We know that the fourth roots of unity are 1, -1, i, and $-i$. Each of these is also an 8th root of unity. Now one additional 8th root of unity is given by $\omega = \cos \pi/4 + i \sin \pi/4 = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$. Also, $-\omega$ is an 8th root of unity, as are the complex conjugate of ω and $-\omega$. So, we can express the four non-real 8th roots of unity as $\pm\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)$ and $\pm\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$.

5. For $n =$ (★)6 find the P matrix for the $n \times n$ W and also find P^{-1}.

   We know that the 6th roots of unity are the powers of by $\omega = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and that $\omega^2 = -\cos \pi/3 + i \sin \pi/3$, $\omega^3 = -1$, $\omega^4 = \overline{\omega^2}$, and $\omega^5 = \overline{\omega}$. We also know that the powers of ω can be reduced modulo 6. This shows that

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & -1 & \overline{\omega^2} & \overline{\omega} \\ 1 & \omega^2 & \overline{\omega^2} & 1 & \omega^2 & \overline{\omega^2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \overline{\omega^2} & \omega^2 & 1 & \overline{\omega^2} & \omega^2 \\ 1 & \overline{\omega} & \overline{\omega^2} & -1 & \omega^2 & \omega \end{bmatrix}.$$

   We also know that $P^{-1} = \frac{1}{6}\overline{P}$.

6. (★) Let $C = \begin{bmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{4} \\ \sqrt[3]{4} & 1 & \sqrt[3]{2} \\ \sqrt[3]{2} & \sqrt[3]{4} & 1 \end{bmatrix}$. Find the eigenvalues and characteristic polynomial of *C*.

Here, $C = q(W)$ where $q(t) = 1 + \sqrt[3]{2}\, t + \sqrt[3]{4}\, t^2$. We know that the eigenvalues of the 3×3 W are the cube roots of unity: 1, ω, and $\overline{\omega}$, where $\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$. The eigenvalues of C are given by

$q(1) = 1 + \sqrt[3]{2} + \sqrt[3]{4}$,

$q(\omega) = 1 + \sqrt[3]{2}\ \omega + \sqrt[3]{4}\ \omega^2 = 1 + \sqrt[3]{2}\ \omega + \sqrt[3]{4}\ \overline{\omega}$, and

$q(\overline{\omega}) = \overline{q(\omega)}$.

For the second of these we compute

$$q(\omega) = 1 + \sqrt[3]{2} \cdot \frac{1}{2}\left(-1 + i\sqrt{3}\right) + \sqrt[3]{4} \cdot \frac{1}{2}\left(-1 - i\sqrt{3}\right)$$
$$= 1 - \frac{\sqrt[3]{2}}{2} - \frac{\sqrt[3]{4}}{2} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{2} - \sqrt[3]{4}\right).$$

The third eigenvalue is then

$$\overline{q(\omega)} = 1 - \frac{\sqrt[3]{2}}{2} - \frac{\sqrt[3]{4}}{2} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{2} - \sqrt[3]{4}\right).$$

Although we could use these three eigenvalues to compute the characteristic polynomial, it is probably more direct to use the determinant formulation:

$$p(t) = \det\begin{bmatrix} t-1 & -\sqrt[3]{2} & -\sqrt[3]{4} \\ -\sqrt[3]{4} & t-1 & -\sqrt[3]{2} \\ -\sqrt[3]{2} & -\sqrt[3]{4} & t-1 \end{bmatrix}.$$

Because this matrix is 3×3, we can compute the determinant using products along diagonal lines, adding the products that go from upper-left to lower-right, and subtracting the products from lower-left to upper-right . That gives

$$p(t) = (t-1)^3 - 2 - 4 - (t-1)(2 + 2 + 2)$$
$$= (t-1)^3 - 6 - 6(t-1).$$

To complete the calculation, expand $(t - 1)^3$:

$$p(t) = t^3 - 3t^2 + 3\mathrm{t} - \mathbf{1} - 6 - 6t + 6$$
$$= t^3 - 3t^2 - 3\mathrm{t} - 1.$$

This example shows that an innocent looking cubic (like $p(t)$) can have very complicated looking roots.