Day 6 Fri 2/2/2018

Collect Formal Prob Set 1,
Take questions on this or any other material.

New Topic: Eigenvectors and eigenvalues.

1. Brief review of determinants
   a. Overview: function from $n \times n$ matrices into $\mathbb{C}$ (or $\mathbb{R}$, or $\mathbb{Z}$, …)
   b. Computation by minors; Definition for 2×2 case.
   c. Key property: $\det(A) \neq 0$ iff A is invertible
   d. Key property: $\det(AB) = \det(A)\det(B)$ (doesn't work for addition/subtraction or scalar multiplication). But you *can* factor a constant out of any row or column.
   e. Key property: If A is triangular, $\det(A)$ = product of diagonal entries
   f. Add'l property: $\det(A) = \det(A^T)$
   g. Add'l property: det is a linear function of the rows of a matrix, likewise cols
   h. Add'l property: det is a polynomial function of the n^2 entries of a matrix.
   i. Add'l property: Cramer's Rule
   j. Add'l property: Vandermonde determinant

2. Definition of eigenvalue; eigenvector
   a. Key idea: λ is an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvector iff $A\mathbf{v} = \lambda\mathbf{v}$
   b. Technical point: For every scalar λ, $A\mathbf{v} = \lambda\mathbf{v}$ holds for the vector $\mathbf{v} = \mathbf{0}$. But that is kind of a trivial instance. We only consider λ to be an eigenvalue if it has a nontrivial eigenvector.
   c. Definition of eigenvalue: λ is an eigenvalue of square matrix A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
   d. Definition of eigenvector: $\mathbf{v}$ is an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some eigenvalue λ of A.
   e. Every eigenvalue has the zero vector as an eigenvector.
   f. A zero eigenvalue is possible, iff there are nontrivial solutions to $A\mathbf{v} = \mathbf{0}$.

3. Finding eigenvalues and eigenvectors
   a. Find eigenvalues first, then eigenvectors
   b. Consider the equation $A\mathbf{v} = \lambda\mathbf{v}$ considering both λ and $\mathbf{v}$ as unknowns.
   c. Goal: find λ and $\mathbf{v}$ with nonzero $\mathbf{v}$.
   d. Reformulation: $A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow A\mathbf{v} = \lambda I\mathbf{v} \Leftrightarrow A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0} \Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$. So we want to choose λ for which $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has nontrivial solutions, and then find all the solutions to that system.
   e. Theorem: a linear homogeneous system of n equations in n unknowns has a nontrivial solution iff the determinant of the coefficient matrix is 0.
   f. Conclusion: λ is an eigenvalue iff $\det(A - \lambda I) = 0$ iff $\det(\lambda I - A) = 0$

g. Example: $A = \begin{bmatrix} 1 & -15 & -3 \\ -6 & 28 & 6 \\ 30 & -150 & -32 \end{bmatrix}$. Equation det($\lambda I - A$) = 0 reduces to $(\lambda - 1)(\lambda + 2)^2 = 0$. This shows the eigenvalues are 1, -2, -2. To find eigenvectors for λ = 1, we have to solve the homogeneous system $(I - A)\mathbf{v} = \mathbf{0}$. So find the reduced row echelon form for $I - A$. This shows that the eigenvectors are expressed in the form c $[1\ \ -2\ \ 10]^T$ where c can be any scalar. For $\lambda = -2$, we have to solve the homogeneous system $(-2I - A)\mathbf{v} = \mathbf{0}$. , or equivalently for $(A + 2I)\mathbf{v} = \mathbf{0}$. So this time find the reduced row echelon form for $A + 2I$. This time there are *two* free variables and the general solution has the form $\mathbf{v} = b[5\ \ 1\ \ 0]^T + c[1\ \ 0\ \ 1]^T$, where b and c can be any scalars. Notice that this time we actually have two linearly independent eigenvectors (and all their combinations). It is not a coincidence that this happens for the repeated eigenvalue. A general theorem says that the maximum number of independent eigenvectors for any eigenvalue cannot exceed the multiplicity of the eigenvalue as a root of the equation det($\lambda I - A$) = 0. In this example the number of independent eigenvectors equals the multiplicity. But in other cases it can be strictly less than the multiplicity. See next example.

h. characteristic polynomial: det($\lambda I - A$); characteristic equation: det($\lambda I - A$) = 0. Eigenvalues are roots of the characteristic polynomial, which has degree n for an $n \times n$ matrix. The multiplicity of the eigenvalue is its multiplicity as a root. There can be at most n distinct eigenvalues for an $n \times n$ matrix.

i. Example: $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$. Here, using what we know about determinants of triangular matrices, we can see by inspection that the characteristic polynomial is $(\lambda - 5)^3$. So there is only one eigenvalue, 5, and it has multiplicity 3. To find eigenvectors, we have to solve the homogeneous system $(5I - A)\mathbf{v} = \mathbf{0}$ (or equivalently, $(A - 5I)\mathbf{v} = \mathbf{0}$,), and this time the matrix $A - 5I$ is already in reduced row echelon form. It tells us that the eigenvectors can all be expressed in the form $c[1\ \ 0\ \ 0]^T$. There is only one independent eigenvector even though the multiplicity of the eigenvalue is 3.

4. Time Permitting: begin discussion of diagonalization.

End of Day