Day 27 Tue 4/24/2018

Course Grade determination:

W/O final exam: each prob set = 25% and exam 1 = 25%

W Final exam, each prob set = 20% and each exam = 20%

this is same as (.80)(score w/o final) + (.20)(final exam score)

Final exam will be cumulative, but will emphasize material since 1st exam

Finish proof of the JCF of monorootic companion matrix.
Start at point 6h from previous class.

Start the derivation of a matrix formula for the sum of the rth powers

1. intro: formula for sums of rth powers of integers 1 through k
   a. r = 1 and 2
   b. definition of $S_k^{\langle r \rangle} = \sum_{j=1}^{k} j^r = \sum_{j=0}^{k} j^r$
   c. r = 0
   d. r = 3 and greater
   e. general formula provided by matrix analysis

2. Difference equation approach
   a. Example: compute $(L-1)S_k^{\langle 2 \rangle} = (k+1)^2$
   b. This is an *inhomogeneous* difference equation
   c. Show that $(L-1)^4 S_k^{\langle 2 \rangle} = 0$ so we can analyze with monorootic case results:
   d. $S_k^{\langle 2 \rangle} = [1 \quad 0 \quad 0 \quad 0] C^k \left[S_0^{\langle 2 \rangle} \quad S_1^{\langle 2 \rangle} \quad S_2^{\langle 2 \rangle} \quad S_3^{\langle 2 \rangle} \right]^T$
      $= [1 \quad 0 \quad 0 \quad 0] C^k [\ 0 \quad 1 \quad 5 \quad 14]^T$
      where *C* is a 4 x 4 monorootic companion matrix with eigenvalue 1, and thus

$$C^k = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}^k \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} \\ & 1 & \binom{k}{1} & \binom{k}{2} \\ & & 1 & \binom{k}{1} \\ & & & 1 \end{bmatrix}$$

e. So $$S_k^{\langle 3\rangle} = [1\ \ 0\ \ 0\ \ 0]\begin{bmatrix}1&&&\\1&1&&\\1&2&1&\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&\binom{k}{1}&\binom{k}{2}&\binom{k}{3}\\&1&\binom{k}{1}&\binom{k}{2}\\&&1&\binom{k}{1}\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\-1&1&&\\1&-2&1&\\-1&3&-3&1\end{bmatrix}\begin{bmatrix}0\\1\\5\\14\end{bmatrix}$$

$$= [1\ \ 0\ \ 0\ \ 0]\begin{bmatrix}1&\binom{k}{1}&\binom{k}{2}&\binom{k}{3}\\&1&\binom{k}{1}&\binom{k}{2}\\&&1&\binom{k}{1}\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\-1&1&&\\1&-2&1&\\-1&3&-3&1\end{bmatrix}\begin{bmatrix}0\\1\\5\\14\end{bmatrix}$$

$$= \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \ \binom{k}{3}\right]\begin{bmatrix}1&&&\\-1&1&&\\1&-2&1&\\-1&3&-3&1\end{bmatrix}\begin{bmatrix}0\\1\\5\\14\end{bmatrix}$$

$$= \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \ \binom{k}{3}\right]\begin{bmatrix}0\\1\\3\\2\end{bmatrix} = \binom{k}{1} + 3\binom{k}{2} + 2\binom{k}{3}$$

3. General case
   a. $(L-1)S_k^{\langle r\rangle} = (k+1)^r$ is a polynomial of degree r
   b. Proposition: if $a_k = f(k)$ where f is a polynomial of degree r then $(L-1)^{r+1}a_k = 0$. Proof by induction.
   c. $(L-1)^{r+2}S_k^{\langle r\rangle} = 0$
   d. solution is $$S_k^{\langle r\rangle} = \Big[\underbrace{1\ \ 0\ \ 0\ \cdots\ 0}_{r+2}\Big]\, P_{r+2}(1)\cdot(I+N)^k\cdot P_{r+2}(-1)\begin{bmatrix}S_0^{\langle r\rangle}\\S_1^{\langle r\rangle}\\\vdots\\S_{r+1}^{\langle r\rangle}\end{bmatrix}$$
      where $P_{r+2}(1)$ is an $(r+2)\times(r+2)$ Pascal's triangle matrix, and $P_{r+2}(-1)$ is an $(r+2)\times(r+2)$ alternating sign Pascal's triangle matrix
   e. Multiply together the first three factors in the solution equation. The result is $\left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]$.
   f. Conclusion: $$S_k^{\langle r\rangle} = \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]\cdot P_{r+2}(-1)\begin{bmatrix}S_0^{\langle r\rangle}\\S_1^{\langle r\rangle}\\\vdots\\S_{r+1}^{\langle r\rangle}\end{bmatrix}$$

4. Further simplification
   a. note that $$\begin{bmatrix}S_0^{\langle r\rangle}\\S_1^{\langle r\rangle}\\\vdots\\S_{r+1}^{\langle r\rangle}\end{bmatrix} = \begin{bmatrix}1&&&\\1&1&&\\\vdots&\vdots&\ddots&\\1&1&\cdots&1\end{bmatrix}\begin{bmatrix}0^r\\1^r\\\vdots\\(r+1)^r\end{bmatrix}$$
   b. Conclusion: $$S_k^{\langle r\rangle} = \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]\cdot P_{r+2}(-1)\begin{bmatrix}1&&&\\1&1&&\\\vdots&\vdots&\ddots&\\1&1&\cdots&1\end{bmatrix}\begin{bmatrix}0^r\\1^r\\\vdots\\(r+1)^r\end{bmatrix}$$

c. Question: what is $P_{r+2}(-1)\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}$

d. Look at some examples with freemat.

e. The pattern that emerges is that $P_{r+2}(-1)\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix} = \left[\begin{array}{c|c} 1 & 0\ \ 0\ \cdots\ 0 \\ \hline 0 & \\ 0 & P_{r+1}(-1) \\ \vdots & \\ 0 & \end{array}\right]$

f. $S_k^{\langle r\rangle} = \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \ \cdots\ \ \binom{k}{r+1}\right]\left[\begin{array}{c|c} 1 & 0\ \ 0\ \cdots\ 0 \\ \hline 0 & \\ 0 & P_{r+1}(-1) \\ \vdots & \\ 0 & \end{array}\right]\begin{bmatrix}0^r \\ 1^r \\ \vdots \\ (r+1)^r\end{bmatrix}$

g. Note that the first twuo factors produce a row vector, the first entry of which is multiplied by the 0^r in the third factor, and that has no effect on the final result. If we focus on the all the rest of the terms, we find this result:

$$S_k^{\langle r\rangle} = \left[\binom{k}{1}\ \ \binom{k}{2}\ \ \cdots\ \ \binom{k}{r+1}\right]\cdot P_{r+1}(-1)\cdot\begin{bmatrix}1^r \\ \vdots \\ (r+1)^r\end{bmatrix}$$

5. Example: sums of fourth powers.

End of Day