Day 28 Fri 4/27/2018

Announce plans for final exam period. If all opt out of the final, I will hold a final class meeting during the final exam exam period where students can ask questions about any topics from the course, and where I will present some additional material.

Complete discussion of sums of powers of integers.

1. General case
   a. $(L-1)S_k^{\langle r\rangle} = (k+1)^r$ is a polynomial of degree r
   b. Proposition: if $a_k = f(k)$ where f is a polynomial of degree r then $(L-1)^{r+1}a_k = 0$. Proof by induction.
   c. Proposition: Let $s_k = f(0) + f(1) + \cdots + f(k)$ where f is a polynomial of degree r then $(L-1)^{r+2}s_k = 0$.
      Proof: $(L-1)\{s_k\} = \{s_{k+1} - s_k\} = \{f(k+1)\}$. But $f(k+1)$ is a polynomial in k of degree r, so by the preceding proposition, $(L-1)^{r+1}\{f(k+1)\} = 0$. Therefore
      $$\begin{aligned}(L-1)^{r+2}\{s_k\} &= (L-1)^{r+1}(L-1)\{s_k\}\\ &= (L-1)^{r+1}\{f(k+1)\} = 0.\end{aligned}$$
   d. $(L-1)^{r+2}S_k^{\langle r\rangle} = 0$ This is a special case of the preceding proposition.
   e. solution is $$S_k^{\langle r\rangle} = \Big[\underbrace{1\ \ 0\ \ 0\ \cdots\ 0}_{r+2}\Big] P_{r+2}(1)\cdot(I+N)^k\cdot P_{r+2}(-1)\begin{bmatrix} S_0^{\langle r\rangle}\\ S_1^{\langle r\rangle}\\ \vdots\\ S_{r+1}^{\langle r\rangle}\end{bmatrix}$$
      where $P_{r+2}(1)$ is an $(r+2)\times(r+2)$ Pascal's triangle matrix, and $P_{r+2}(-1)$ is an $(r+2)\times(r+2)$ alternating sign Pascal's triangle matrix
   f. Multiply together the first three factors in the solution equation. The result is $\left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]$.
   g. Conclusion: $$S_k^{\langle r\rangle} = \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]\cdot P_{r+2}(-1)\begin{bmatrix} S_0^{\langle r\rangle}\\ S_1^{\langle r\rangle}\\ \vdots\\ S_{r+1}^{\langle r\rangle}\end{bmatrix}$$
2. Further simplification
   a. note that $$\begin{bmatrix} S_0^{\langle r\rangle}\\ S_1^{\langle r\rangle}\\ \vdots\\ S_{r+1}^{\langle r\rangle}\end{bmatrix} = \begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix}0^r\\ 1^r\\ \vdots\\ (r+1)^r\end{bmatrix}$$
   b. Conclusion: $$S_k^{\langle r\rangle} = \left[1\ \ \binom{k}{1}\ \ \binom{k}{2}\ \cdots\ \binom{k}{r+1}\right]\cdot P_{r+2}(-1)\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix}0^r\\ 1^r\\ \vdots\\ (r+1)^r\end{bmatrix}$$

c. Question: what is $P_{r+2}(-1)\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}$

d. Look at some examples with freemat.

e. The pattern that emerges is that $P_{r+2}(-1)\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix} = \left[\begin{array}{c|c} 1 & 0\ \ 0\ \ \cdots\ \ 0 \\ \hline 0 & \\ 0 & P_{r+1}(-1) \\ \vdots & \\ 0 & \end{array}\right]$

f. $S_k^{\langle r\rangle} = \begin{bmatrix}1 & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{r+1}\end{bmatrix}\left[\begin{array}{c|c} 1 & 0\ \ 0\ \ \cdots\ \ 0 \\ \hline 0 & \\ 0 & P_{r+1}(-1) \\ \vdots & \\ 0 & \end{array}\right]\begin{bmatrix}0^r \\ 1^r \\ \vdots \\ (r+1)^r\end{bmatrix}$

g. Note that the first two factors produce a row vector, the first entry of which is multiplied by the 0^r in the third factor, and that has no effect on the final result. If we focus on the all the rest of the terms, we find this result:

$$S_k^{\langle r\rangle} = \begin{bmatrix}\binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{r+1}\end{bmatrix} \cdot P_{r+1}(-1) \cdot \begin{bmatrix}1^r \\ \vdots \\ (r+1)^r\end{bmatrix}$$

h. Actually, using the same logic on the sequence $s_k = f(0) + f(1) + \cdots + f(k)$ discussed earlier, we derive the analogous solution. For any polynomial $f(k)$ of degree r:

$$f(1) + \cdots + f(k) = \begin{bmatrix}\binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{r+1}\end{bmatrix} \cdot P_{r+1}(-1) \cdot \begin{bmatrix}f(1) \\ \vdots \\ f(r+1)\end{bmatrix}$$

3. Example: sums of fourth powers.

Additional Class Time: One or more of the following

1. Review for those taking the final exam
2. Discuss solutions to the third formal problem set
3. Start on a new power point presentation (check the class handouts webpage)

End of Day