Day 21 Tue 4/3/2018

1. Review solution of difference equation with initial vector $\mathbf{e}_n$.
   a. Vectorization: $\mathbf{v}_k$ = k^{th} window vector, C is a companion matrix, difference equation becomes $\mathbf{v}_{k+1} = C\mathbf{v}_k$ so solution is $\mathbf{v}_k = C^k\mathbf{v}_0$.
   b. Characteristic polynomial of C is $p(t)$
   c. Distinct roots case: C diagonalizable iff roots of p are distinct
   d. If the (distinct) roots are $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n,$ and the initial terms given by $a_0 = a_1 = \cdots = a_{n-2} = 0,\ \ a_{n-1} = 1$, then
   $$a_k = \frac{\lambda_1^k}{p'(\lambda_1)} + \frac{\lambda_2^k}{p'(\lambda_2)} + \cdots + \frac{\lambda_n^k}{p'(\lambda_n)} = \sum_{j=1}^{n} \frac{\lambda_j^k}{p'(\lambda_j)}$$
   e. Two examples from end of Companion Matrices Lecture1 slide show.

Example: Fibonacci Numbers

- $a_{k+2} = a_k + a_{k+1}$
- $p(t) = t^2 - t - 1$
- Roots: $\phi = \dfrac{1+\sqrt{5}}{2}$; $\overline{\phi} = \dfrac{1-\sqrt{5}}{2}$
- $p'(t) = 2t - 1$
- $p'(\phi) = \sqrt{5}$; $p'(\overline{\phi}) = -\sqrt{5}$
- $a_k = F_k = \dfrac{1}{\sqrt{5}}\left(\phi^k - \overline{\phi}^k\right)$

Another Example

- $a_{k+4} = -a_k - a_{k+1} + 2a_{k+2} + a_{k+1}$
  $= (-1, -1, 2, 1) \cdot (a_k, a_{k+1}, a_{k+2}, a_{k+3})$
- $(a_0, a_1, a_2, a_3) = (0, 0, 0, 1)$
- First few terms: $0, 0, 0, 1, 1, 3, 4, 8, 12, 21, 33, \cdots$

Solution

- $p(t) = t^4 - t^3 - 2t^2 + t + 1 = (t^2-1)(t^2-t-1)$
- Roots: $1, -1, \phi, \overline{\phi} \quad = \lambda_i;\ i = 1,2,3,4$
- $p'(t) = 4t^3 - 3t^2 - 4t + 1 = \sum_{i=1}^{4} \prod_{j \neq i} (t - \lambda_j)$
- $p'(1) = -2,\ p'(-1) = -2$
- $p'(\phi) = (\phi - 1)(\phi + 1)(\phi - \overline{\phi}) = (\phi^2 - 1)\sqrt{5} = \phi\sqrt{5}$
- $p'(\overline{\phi}) = -\overline{\phi}\sqrt{5},$

$$a_k = \frac{1}{-2} + \frac{(-1)^k}{-2} + \frac{\phi^k}{\phi\sqrt{5}} - \frac{\overline{\phi}^k}{\overline{\phi}\sqrt{5}} = \begin{cases} F_{k-1} - 1 & \text{for even } k \\ F_{k-1} & \text{for odd } k \end{cases}$$

2. Linear difference equations using lag operator L
   a. Definition of lag operator, aka *Left Shift* operator
   $$L(\{a_k\}_{k=0}^{\infty}) = \{a_{k+1}\}_{k=0}^{\infty} = \{a_m\}_{m=1}^{\infty}$$
   b. This is a linear transformation on the vector space $\mathbb{R}^\infty$ (or $\mathbb{C}^\infty$) of infinite sequences. We can think of those as infinitely long row or column vectors.
   c. Powers and polynomials in L are also linear.
   d. A scalar difference equation can be expressed in the form $p(L)\{a_k\} = \{0\}$.
   e. Example: Fibonacci difference equation $a_{k+2} = a_{k+1} + a_k$ becomes $(L^2 - L - 1)\{a_k\} = \{0\}$.
   f. Characteristic polynomial is $p(t)$ for the same p as in the equation with $p(L)$. This is the same as the characteristic polynomial for the companion matrix C in the vectorized difference equation.
   g. Because L is a linear transformation, so is $p(L)$, and the equation $p(L)\{a_k\} = \{0\}$ is a homogeneous linear equation. It's solutions are a subspace of $\mathbb{C}^\infty$, namely the null space of the linear transformation $p(L)$.
   h. As a subspace, it is closed under addition and scalar multiplication. That means linear combinations of solution sequences are themselves solution sequences.
   i. In the terminology of physics and engineering, the superposition principle applies: we can obtain new solutions by taking linear combinations of known solutions.

3. Example: Fibonacci difference equation
   a. The standard Fibonacci numbers $\{F_k\}_{k=0}^{\infty} = (0,1,1,2,3,5,8,\ldots)$ is one solution to the equation $(L^2 - L - 1)\{a_k\} = \{0\}$.
   b. What if we begin with (1,0) instead of (0,1)? That produces the sequence (1,0,1,1,2,3,5,8,…). Call this $\{G_k\}_{k=0}^{\infty}$.
   c. Any linear combination of these two sequences also satisfies the same difference equation.
   d. For example $5\{G_k\} + 3\{F_k\} = 5(1,0,1,1,\cdots) + 3(0,1,1,2,\cdots) = (5,3,8,11,\cdots)$
   e. Notice that this gives us a solution to the difference equation with initial terms (5,3).
   f. In a similar way, we can express the solution to the difference equation with ANY initial terms (u, v) as $u\{G_k\} + v\{F_k\}$.
   g. Also, notice that for $k > 0$, $G_k = F_{k-1}$. So the with initial terms u and v, the solution sequence has k^{th} term $uF_{k-1} + vF_k$ (for $k > 0$).
   h. In item d above, the k^{th} term of $(5,3,8,11,\cdots)$ is $a_k = 5F_{k-1} + 3F_k$ (for $k > 0$).

4. Backwards recursion
   a. The Fibonacci recursion that allows us to compute any term as combination of the two preceding terms, can also be used to find any term as a combination of the two *following* terms.
   b. For example, we can find F_7 using $F_7 = F_5 + F_6$, but we could just as well find $F_5 = F_7 - F_6$. In particular, $F_{-1} = F_1 - F_0 = 1$.
   c. With that definition, the equation $a_k = 5F_{k-1} + 3F_k$ holds even for $k = 0$.
   d. The matrix-vector formulation for a general difference equation, $\mathbf{a}_{k+1} = C\mathbf{a}_k$, can be reversed by writing $\mathbf{a}_k = C^{-1}\mathbf{a}_{k+1}$, as long as the companion matrix is invertible.
   e. Alternatively, we can find the matrix representation of the backward recursion using row operation concepts. In this way we can derive a simple formula for the inverse of a companion matrix. These ideas were explored in exercises 5 and 10 of **Companion Matrices Exercise Set 1**.

5. A different approach
   a. We know that the standard Fibonacci numbers satisfy the difference equation $(L^2 - L - 1)\{a_k\} = \{0\}$.
   b. Apply the L transformation to both sides, to find $L(L^2 - L - 1)\{a_k\} = L\{0\} = \{0\}$.
   c. Now observe that L commutes with itself, so $L(L^2 - L - 1) = (L^2 - L - 1)L$.
   d. Therefore $(L^2 - L - 1)L\{a_k\} = \{0\}$.
   e. This shows that $L\{a_k\}$ is another solution to the difference equation.
   f. In words: if $\{a_k\}$ is a solution to a difference equation, so is $L\{a_k\}$.
   g. By repeating this procedure m times we can conclude $L^m\{a_k\}$ is a solution to the same difference equation. Note that $L^m\{a_k\}$ is $\{a_k\}$ with the first m terms deleted. So it is apparent that this sequence will satisfy the same difference equation as $\{a_k\}$.

6. Applying this to the Fibonacci difference equation
   a. The standard Fibonacci numbers $\{F_k\}_{k=0}^{\infty}$ = (0,1,1,2,3,5, …) is one solution to the equation $(L^2 - L - 1)\{a_k\} = \{0\}$.
   b. Also, $L\{F_k\}_{k=0}^{\infty} = \{F_{k+1}\}_{k=0}^{\infty}$ = (1,1,2,3,5,8,…) is another solution.
   c. Any linear combination gives another solution. With coefficients of 6 and 2 we find $6\{F_{k+1}\} + 2\{F_k\} = 6(1,1,2,3,5,\cdots) + 2(0,1,1,2,3\ \cdots) = (6,8,14,22,\cdots)$ is the solution with initial terms 6 and 8.
   d. What if we want initial terms of 4 and -6? Take the coefficients to be *u* and *v*, form $u\{F_{k+1}\} + v\{F_k\} = u(1,1,2,3,5,\cdots) + v(0,1,1,2,3\ \cdots) = (u, u+v, 2u+v, \cdots)$ and then set (*u*, *u*+*v*) = (4, -6).
   e. That gives us two equations in 2 unknowns, and we solve to find *u* = 4 and *v* = -10.
   f. That is, $4(1,1,2,3,5,\cdots) - 10(0,1,1,2,3\ \cdots) = (4,-6,-2,-8,\cdots)$, and the *k*th term of this sequence is $4F_{k+1} - 10F_k$.
   g. In a similar way, for ANY initial terms, we can express the solution to the Fibonacci difference equation as a combination of F_k and F_{k+1}.
   h. Next we will apply this idea to extend our equation for *Generalized Fibonacci* numbers to solve difference equations with ANY initial vector, not just $\boldsymbol{e}_n$.

7. Another example
   a. Reconsider the example
   $a_{k+4} = a_{k+3} + 2a_{k+2} - a_{k+1} - a_k\ ;\ \ a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 1.$
   b. Recall solution sequence is (0, 0, 0, 1, 1, 3, 4, 8, 12, 21, ⋯). Throughout this example, we will refer to this as
   $\{E_k\}_{k=0}^{\infty} = (E_0, E_1, E_2, E_3, \cdots) = (0,0,0,1,1,3,4,8,12,21,\cdots)$.
   c. Also consider $L\{E_k\}_{k=0}^{\infty} = (\ 0,0,1,1,3,4,8,12,21,\cdots)$,
   $L^2\{E_k\}_{k=0}^{\infty} = (\ 0,1,1,3,4,8,12,21,\cdots)$, and
   $L^3\{E_k\}_{k=0}^{\infty} = (\ 1,1,3,4,8,12,21,\cdots)$.
   d. Now suppose we want a sequence with the same difference equation but a different initial sequence. For concreteness, say we want $a_0 = 10,\ a_1 = 13,\ a_2 = 17,$ and $a_3 = 19$. We will try to obtain these conditions by forming a linear combination of our four known solutions, $\{E_k\}, L\{E_k\}, L^2\{E_k\}, L^3\{E_k\}$, with coefficients $x_0,\ x_1,\ x_2,$ and x_3.
   e. We want to have
   $$x_0\{E_k\} + x_1L\{E_k\} + x_2L^2\{E_k\} + x_3L^3\{E_k\} = (10,13,17,19,30,38,\cdots)\quad .$$
   f. Expand the left side:
   $$\begin{aligned} x_0\{E_k\} &= (0,\ 0,\ 0,\ x_0,\ x_0,\ 3x_0,\ 4x_0,\ 8x_0,\cdots) \\ x_1L\{E_k\} &= (0,\ 0,\ x_1,\ x_1,\ 3x_1,\ 4x_1,\ 8x_1,\ 12x_1,\cdots) \\ x_2L\{E_k\} &= (0,\ x_2,\ x_2, 3x_2,\ 4x_2,\ 8x_2,\ 12x_2,\ 21x_2\cdots) \\ x_3L\{E_k\} &= (x_3,\ x_3, 3x_3, 4x_3,\ 8x_3,\ 12x_3,\ 21x_3,\ 33x_3\cdots) \end{aligned}$$

g. Adding these up is supposed to produce $(10, 13, 17, 19, 30, 38, \cdots)$, so looking just at the first four terms of each sequence we obtain these equations:

$$\begin{aligned} x_3 &= 10 \\ x_2 + \; x_3 &= 13 \\ x_1 + \; x_2 + 3x_3 &= 17 \\ x_0 + x_1 + 3x_2 + 4x_3 &= 19 \end{aligned}$$

h. This is a triangular system of linear equations and we can solve it by inspection:

$$\begin{aligned} x_3 &= 10 \\ x_2 + 10 &= 13, \text{ so } \; x_2 = 3 \\ x_1 + 3 + 30 &= 17, \text{ so } \; x_1 = -16 \\ x_0 - 16 + 9 + 40 &= 19, \text{ so } \; x_0 = -14 \end{aligned}$$

i. This shows that

$$-14\{E_k\} - 16L\{E_k\} + 3L^2\{E_k\} + 10L^3\{E_k\} = (10, 13, 17, 19, \; \cdots)$$

so the first four terms are correct. And, since all the sequences on the left obey the given difference equation, so does their sum, proving that all the terms after the 19 will also be correct.

j. This gives us the equation $a_k = -14E_k - 16E_{k+1} + 3E_{k+2} + 10E_{k+3}$ for the solution to the difference equation with the specified initial terms.

k. Let's go further. We know that $E_k = \frac{\lambda_1^k}{p\prime(\lambda_1)} + \frac{\lambda_2^k}{p\prime(\lambda_2)} + \frac{\lambda_3^k}{p\prime(\lambda_3)} + \frac{\lambda_4^k}{p\prime(\lambda_4)} = \sum_{j=1}^4 \frac{\lambda_j^k}{p\prime(\lambda_j)}$.

Similarly $E_{k+1} = \frac{\lambda_1^{k+1}}{p\prime(\lambda_1)} + \frac{\lambda_2^{k+1}}{p\prime(\lambda_2)} + \frac{\lambda_3^{k+1}}{p\prime(\lambda_3)} + \frac{\lambda_4^{k+1}}{p\prime(\lambda_4)} = \sum_{j=1}^4 \frac{\lambda_j^{k+1}}{p\prime(\lambda_j)}$. So, we can write

$$\begin{aligned} a_k &= -14\sum_{j=1}^4 \frac{\lambda_j^k}{p\prime(\lambda_j)} - 16\sum_{j=1}^4 \lambda_j \frac{\lambda_j^k}{p\prime(\lambda_j)} + 3\sum_{j=1}^4 \lambda_j^2 \frac{\lambda_j^k}{p\prime(\lambda_j)} + 10\sum_{j=1}^4 \lambda_j^3 \frac{\lambda_j^k}{p\prime(\lambda_j)} \\ &= \sum_{j=1}^4 -14\frac{\lambda_j^k}{p\prime(\lambda_j)} + \sum_{j=1}^4 -16\lambda_j \frac{\lambda_j^k}{p\prime(\lambda_j)} + \sum_{j=1}^4 3\lambda_j^2 \frac{\lambda_j^k}{p\prime(\lambda_j)} + \sum_{j=1}^4 10\lambda_j^3 \frac{\lambda_j^k}{p\prime(\lambda_j)} \\ &= \sum_{j=1}^4 \left(-14 - 16\lambda_j + 3\lambda_j^2 + 10\lambda_j^3\right)\frac{\lambda_j^k}{p\prime(\lambda_j)} = \sum_{j=1}^4 q(\lambda_j)\frac{\lambda_j^k}{p\prime(\lambda_j)} \end{aligned}$$

where $q(t) = -14 - 16t + 3t^2 + 10t^3$.

l. Perhaps more appealingly, $a_k = \sum_{j=1}^4 \frac{q(\lambda_j)}{p\prime(\lambda_j)}\lambda_j^k$.

m. We are going to see next that there is a simple the polynomial $q(t)$ is related to $p(t)$ in a simple way. As a result, given any n^{th} order difference equation and any initial vector, as long as the characteristic polynomial has distinct roots, we will be able to determine both $q(t)$ and $p'(t)$, leading to the equation $a_k = \sum_{j=1}^n \frac{q(\lambda_j)}{p\prime(\lambda_j)}\lambda_j^k$ for the solution.

HOMEWORK: Exercises on Lag Operator and Difference Equations, 1-5

End of Day