Circulant Matrices

- Example, 3×3

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

- Example, 4×4

$$\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

- Circulants arise naturally in many applications, including signal processing and graph theory, have many interesting properties

Eigenvalues and Eigenvectors

- Consider a circulant with first row $[c_0 \;\; c_1 \;\; c_2 \;\; \cdots \;\; c_{n-1}]$
- Define $q(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$
- Eigenvalues are $q(\omega)$ where ω is an nth root of unity
- Eigenvectors have form $v_\omega = [1 \;\; \omega \;\; \omega^2 \;\; \cdots \;\; \omega^{n-1}]^T$
- Eigenvalues can be read off by inspection

As we have seen in class, the circulant matrix here is $q(W)$ where W is the n x n permutation matrix defined by moving the top row of I to the bottom. The eigenvalues of W are the roots of $t^n = 1$. These are the nth roots of unity. They are given by ω^k for $k = 0, 1, 2, \ldots, n-1$, where $\omega = \cos(2\pi / n) + i \sin(2\pi / n)$. The eigenvectors of the circulant matrix are the same as the eigenvectors of W. On these slides, the symbol ω is used for *any* nth root of unity, meaning any power of the specific root $\cos(2\pi / n) + i \sin(2\pi / n)$. Think of ω here as the same as α in our class.

Example

$$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$\omega \in \{1, -1, i, -i\}$$

$$q(t) = 1 + 2t + t^2 + 3t^3$$

eigenvalues	eigenvectors
$q(1) = 6$	$v_1 = (1, 1, 1, 1)$
$q(-1) = -3$	$v_{-1} = (1, -1, 1, -1)$
$q(i) = -i$	$v_i = (1, i, -1, -i)$
$q(-i) = i$	$v_{-i} = (1, -i, -1, i)$

Note: characteristic polynomial of C is $p(t) = t^4 - 4t^3 - 20t^2 - 4t - 21$

Another Example

$$C = \begin{bmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{4} \\ \sqrt[3]{4} & 1 & \sqrt[3]{2} \\ \sqrt[3]{2} & \sqrt[3]{4} & 1 \end{bmatrix}$$

$$q(t) = 1 + \sqrt[3]{2}t + \sqrt[3]{4}t^2$$

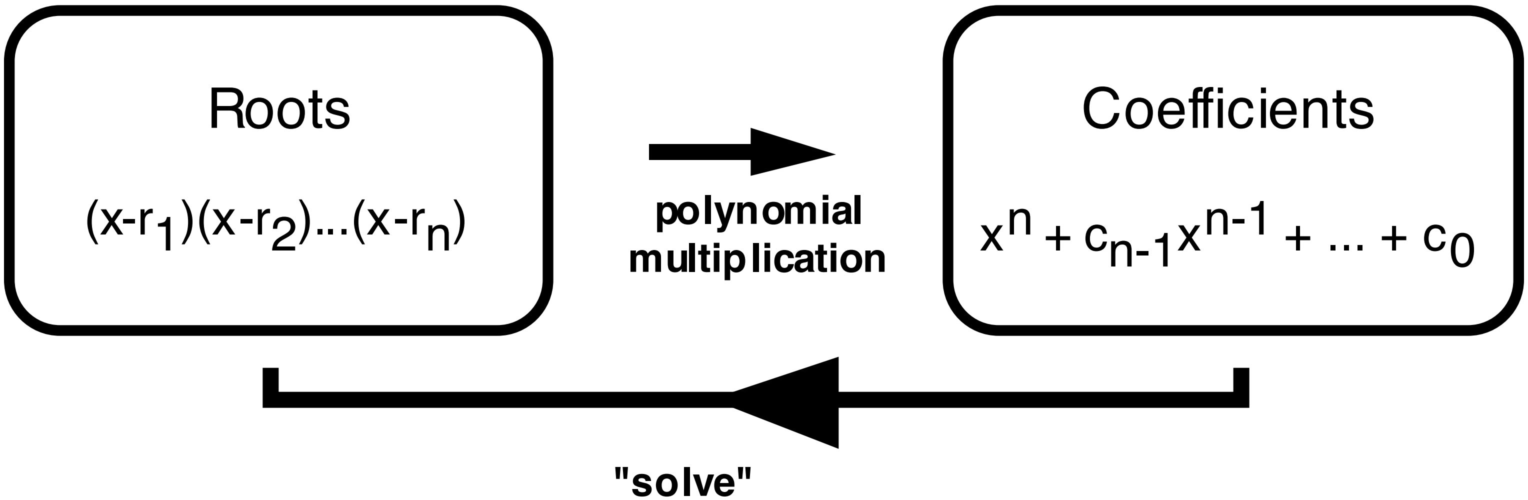
$$\omega \in \left\{ 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2} \right\}$$

Eigenvalues: $1 + \sqrt[3]{2} + \sqrt[3]{4}$, $1 - (\frac{1}{2})\sqrt[3]{2} - (\frac{1}{2})\sqrt[3]{4} \pm (\frac{1}{2}) i\sqrt{3}(\sqrt[3]{4} - \sqrt[3]{2})$.
Characteristic polynomial: $p(t) = t^3 - 3t^2 - 3t - 1$.

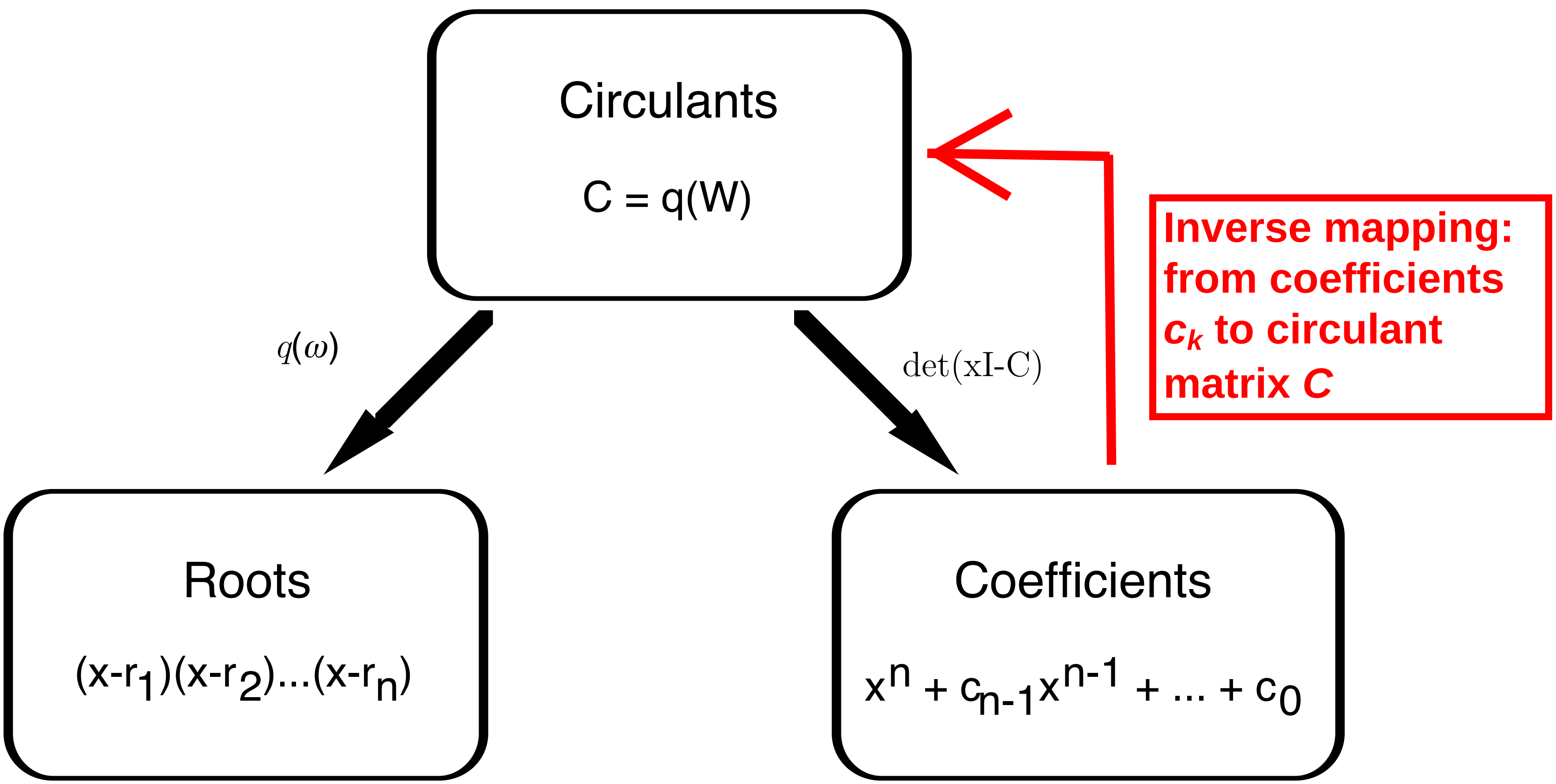
Solving Polynomials with Circulants

- Usual notion of solving a polynomial: given coefficients, find roots
- Circulants give us a rich set of polynomials with known roots
- New approach to solving a polynomial: given p (defined in terms of coefficients), find a circulant matrix $C = q(W)$ for which p is the characterisic polynomial. The eigenvalues $q(\omega)$ of C are then the roots of p

Usual Method



Circulant Method



Example

- $p(t) = t^3 - 3t^2 - 3t - 1$

From an earlier example we know

- p is the characteristic polynomial of $C = \begin{bmatrix} 1 & \sqrt[3]{2} & \sqrt[3]{4} \\ \sqrt[3]{4} & 1 & \sqrt[3]{2} \\ \sqrt[3]{2} & \sqrt[3]{4} & 1 \end{bmatrix}$

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- By inspection, eigenvalues are $q(\omega) = 1 + \omega\sqrt[3]{2} + \omega^2\sqrt[3]{4}$ where ω is a cuberoot of unity

- This gives the roots of p

Finding the right Circulant

- Given monic p of degree n, find a circulant matrix $C(p)$ for which the characteristic polynomial is p
- $C(p) = q(W)$ for an appropriate polynomial q of degree $n-1$
- Existence: if the roots of p are r_k, and the n^{th} roots of unity are ω_k, it suffices to have $q(\omega_k) = r_k$. Existence of q assured by polynomial interpolation theory.
- Polynomial Interpolation: find a polynomial $p(x)$ whose graph goes through n given points (a, b), so that for each point, $p(a) = b$.

Another View

Let $q(x) = q_0 + q_1 x + \cdots + q_{n-1}x^{n-1}$. Let the roots of p be r_k for $1 \leq k \leq n$. Let $\omega = e^{2\pi i/n}$, so that the powers of ω are the n^{th} roots of unity. Then q maps the roots of unity to the r_k if

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{n-1} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}$$

We know that the matrix P in this equation is invertible, and in fact the inverse is given by the $1/n$ times the complex conjugate of P. This shows that the system is solvable for any choice of the r's. Conclusion: given any (monic) degree n polynomial p, there exists a corresponding $n \times n$ circulant matrix C whose characteristic polynomial equals p.

Recap

- Preceding arguments show that **any** monic degree n polynomial can be realized as the characteristic polynomial for some circulant matrix $q(W)$
- We do not know the roots r_k
- Need an alternate way to find q
- Then we can compute the roots by applying q to roots of unity
- Next step: compute a generic circulant characteristic polynomial

- Example: $p(x) = x^3 - 3x^2 + 4x - 12.$
- Want this to be the char polynomial for a 3 × 3 circulant, say

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} = aI + bW + cW^2.$$

- $$\det \begin{bmatrix} x-a & -b & -c \\ -c & x-a & -b \\ -b & -c & x-a \end{bmatrix} = (x-a)^3 - b^3 - c^3 - 3bc(x-a)$$
- We want $(x-a)^3 - b^3 - c^3 - 3bc(x-a) = p(x)$
- Expand and simplify left side:

  $x^3 - 3ax^2 + 3a^2x - a^3 - 3bcx + 3abc - b^3 - c^3$

  $= x^3 - 3ax^2 + 3(a^2 - bc)x + 3abc - a^3 - b^3 - c^3$
- We need this to equal $x^3 - 3x^2 + 4x - 12.$ That requires

  $3a = 3,\ \ 3(a^2 - bc) = 4,$ and $3abc - a^3 - b^3 - c^3 = -12.$

- $3a = 3, \;\; 3(a^2 - bc) = 4,$ and $3abc - a^3 - b^3 - c^3 = -12.$
- $a = 1$, so $3 - 3bc = 4$ and $3bc - 1 - b^3 - c^3 = -12.$
- $3bc = -1$ & $b^3 + c^3 = 10 \Leftrightarrow bc = -1/3$ & $b^3 + c^3 = 10$
- $\Leftrightarrow \;\; b^3c^3 = -1/27$ and $b^3 + c^3 = 10$
- Consider this to be a system whose unknowns are b^3 and c^3.
- In fact, it's a very special system: Find two numbers with a specified sum and product
- We have seen that the solutions are roots to $t^2 - 10t - 1/27.$
- $b^3, \; c^3 = \frac{1}{2}\left(10 \pm \sqrt{100 + 4/27}\right)$

  $$= 5 \pm \sqrt{25 + 1/27} = 5 \pm \sqrt{(25 \cdot 27 + 1)/27}$$
- Note $25 \cdot 27 = (26 - 1)(26 + 1) = 26^2 - 1.$ So…
- $b^3, \; c^3 = 5 \pm \sqrt{26^2/27} = 5 \pm \frac{26}{9}\sqrt{3} = \frac{1}{9}\left(45 \pm 26\sqrt{3}\right)$

- This gives the solutions to our problem:

$$a = 1,\ b = \sqrt[3]{\frac{1}{9}\left(45 + 26\sqrt{3}\right)},\ c = \sqrt[3]{\frac{1}{9}\left(45 - 26\sqrt{3}\right)}$$

- We know that the eigenvalues of the corresponding circulant matrix will be
$a + b + c,\ \ a + \omega b + \bar{\omega} c,$ and $a + \bar{\omega} b + \omega c.$
- In particular, one root is

$$1 + \sqrt[3]{\frac{1}{9}\left(45 + 26\sqrt{3}\right)} + \sqrt[3]{\frac{1}{9}\left(45 - 26\sqrt{3}\right)}$$

- This is indeed a root of the cubic, and is actually equal to 3, though it is not in a very convenient form. Nevertheless, since a calculator computes this as 3, we can verify that 3 is a root by substitution in the original polynomial $x^3 - 3x^2 + 4x - 12$.

- This leads to the factorization

$$x^3 - 3x^2 + 4x - 12 = (x - 3)(x^2 + 4),$$

and hence to the remaining roots $\pm 2i$.