Exercises 1

Note: Exercises marked with (★) have solutions at the end of the handout.

For any polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

1. (★) Prove the sum of the roots is $-a_{n-1} / a_n$

2. Prove the average of the roots of p is equal to the average of the roots of the derivative p'.

3. (★)Prove this: For any polynomial p of degree n with nonzero constant term,
   Rev $p(x) = x^n \, p(1/x)$.

4. Use 1 to prove that the roots of Rev $p(x)$ are the reciprocals of the roots of $p(x)$.

5. (★) Use 2 to prove: if
   $p(x) = a_n x^n + \cdots + a_1 x + a_0$
   and $a_0 \neq 0$, then the sum of the reciprocals of the roots of p is $-a_1 / a_0$

6. Prove that

$$\frac{\text{Rev } p'(x)}{\text{Rev } p(x)} = s_0 + s_1 x + s_2 x^2 + s_3 x^3 + \cdots$$

where s_k is the sum of the k^{th} powers of the roots of p.

Hints: Use (and prove as necessary) the following:

- Rev $f(x) = x^n f(1/x)$ when f is a polynomial of degree n and $f(0) \neq 0$.
- Logarithmic Derivative: $f' / f = (\ln f)'$
- If $f(x) = (x - r)(x - s)(x - t) \cdots$ then $(\ln f(x))' = \frac{1}{x-r} + \frac{1}{x-s} + \frac{1}{x-t} + \cdots$
- Geometric Series: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ and $\frac{1}{x-a} = \frac{-1/a}{1-x/a}$
- But caution: Rev $f'(x) \neq (\text{Rev } f(x))'$

7. (★) Let $p(x) = x^4 + 2x^3 - 5x^2 + 2x + 1.$
   Find the roots, assuming they come in reciprocal pairs (see lecture slide 27).

8. (★) Let $p(x) = x^6 - 7x^5 + 15x^4 - 14x^3 + 15x^2 - 7x + 1.$
   Find the roots using the general reduction method (see lecture slides 30, 31).

Define a polynomial $p(x)$ to anti-palindromic if the reverse polynomial equals $-p(x)$. That is, if $p(x)$ has degree n, then p is anti-palindromic if and only the coefficient of x^k is the opposite of the coefficient of x^{n-k}. So for example, $x^5 + 3x^4 + 7x^3 - 7x^2 - 3x - 1$ is anti-palindromic. The next two problems concern anti-palindromic polynomials.

9. Prove that $p(x)$ is anti-palindromic if and only if

$$p(x) = (x - 1)q(x)$$

where $q(x)$ is palindromic.

(★)10. Show that if r is a non-zero root of an anti-palindromic polynomial so is $1/r$.

Selected Solutions 1

For any polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

1. Prove the sum of the roots is $-a_{n-1} / a_n$.

We proceed by induction. If $n = 1$ then $p(x) = a_1 x + a_0$ and the only root is $-a_0 / a_1$. This is also the sum of the roots, showing that the desired result holds for $n = 1$.

For the induction step, assume that the result holds for any polynomial of degree $n - 1$, and consider a polynomial p of degree n. If the roots are $r_1 \ldots r_n$, then we can write $p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. Now let $q(x) = (x - r_1)(x - r_2)\cdots(x - r_{n-1})$. This has degree $n - 1$, so the induction hypothesis applies. We can write q in descending form as $q(x) = x^{n-1} + b_{n-2}x^{n-2}$ + (lower order terms). By the induction hypothesis, the sum of the roots of q is given by the ratio $-b_{n-2} / 1$, leading to the equation

$$b_{n-2} = -(r_1 + r_2 + \cdots + r_{n-1}).$$

Next, observe that $p(x) = a_n(x - r_n)q(x) = a_n(x - r_n)(x^{n-1} + b_{n-2}x^{n-2} + \text{lower order terms})$. Multiplying out the first few terms of the product shows that $p(x) = a_n(x^n - r_n x^{n-1} + b_{n-2}x^{n-1} + \text{lower order terms})$. Combining like terms we find $p(x) = a_n(x^n + (b_{n-2} - r_n)x^{n-1} + \text{lower order terms})$. Comparing this equation with the definition of $p(x)$ in the problem statement, we see that $a_n(b_{n-2} - r_n) = a_{n-1}$. Therefore, we conclude that $\frac{a_{n-1}}{a_n} = b_{n-2} - r_n = -(r_1 + r_2 + \cdots + r_{n-1}) - r_n$. Multiplying both sides by -1 verifies that

$$-\frac{a_{n-1}}{a_n} = r_1 + r_2 + \cdots + r_{n-1} + r_n.$$

This shows that the sum of the roots of p is $-a_{n-1} / a_n$, completing the induction proof.

3. Prove this: For any polynomial p of degree n with nonzero constant term,
   Rev $p(x) = x^n p(1/x)$.

Solution: Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and observe that Rev $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$. On the other hand, we see that

$$x^n p\left(\frac{1}{x}\right) = x^n p(x^{-1}) = x^n \sum_{k=0}^{n} a_k x^{-k} = \sum_{k=0}^{n} a_k x^{n-k}.$$

This shows that Rev $p(x) = x^n p\left(\frac{1}{x}\right)$, as required.

5. Use 2 to prove: if
$p(x) = a_n x^n + \cdots + a_1 x + a_0$
and $a_0 \neq 0$, then the sum of the reciprocals of the roots of p is $-a_1/a_0$

Solution: Let $p(x) = \sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$ and let $q(x) = \text{Rev}\, p(x) = \sum_{k=0}^{n} a_k x^{n-k}$. We know that the roots of q are nonzero (because its constant term is not 0) and are the reciprocals of the roots of p. Therefore, the sum of the reciprocals of the roots of p is equal to the sum of the roots of q. That in turn is given by $-b_{n-1}/b_n$ where b_{n-1} is the coefficient of x^{n-1} in q and b_n is the coefficient of x^n in q. But our equation for q shows that these coefficients are none other than a_1 and a_0, respectively. Thus the sum of the roots of q is given by $-a_1/a_0$. That completes the proof.

7. Let $p(x) = x^4 + 2x^3 - 5x^2 + 2x + 1.$
Find the roots, assuming they come in reciprocal pairs (see lecture slide 27).

Solution:

A quick mental calculation confirms that neither 1 nor -1 is a root. Therefore, we can assume roots of r, s, and their reciprocals. This means that $p(x)$ factors as

$$\begin{aligned} p(x) &= (x-r)(x-1/r)(x-s)(x-1/s) \\ &= (x^2 - ux + 1)(x^2 - vx + 1) \end{aligned}$$

where $u = r + 1/r$ and $v = s + 1/s$. Now we are led to the equations $u + v = -2$ and $uv = -7$. They tell us that u and v are the roots of the quadratic $x^2 + 2x - 7$, and so are given by $-1 \pm 2\sqrt{2}$. This in turn gives us the factorization

$$p(x) = [x^2 + (1 - 2\sqrt{2})x + 1][x^2 + (1 + 2\sqrt{2})x + 1],$$

so we can find the roots of p by finding the roots of the quadratic factors. For the first factor the roots are

$$\frac{-1 + 2\sqrt{2} \pm i\sqrt{4\sqrt{2} - 5}}{2}$$

while the roots of the second factor are

$$\frac{-1 - 2\sqrt{2} \pm \sqrt{5 + 4\sqrt{2}}}{2}.$$

8. Let $p(x) = x^6 - 7x^5 + 15x^4 - 14x^3 + 15x^2 - 7x + 1.$
Find the roots using the general reduction method.

Solution:

Here, $p(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1$
where the coefficients are $a = -7$, $b = 15$, and $c = -14$.
Therefore, the roots of p satisfy

$$x^3 + ax^2 + bx^1 + c + bx^{-1} + ax^{-2} + x^{-3} = 0.$$

Grouping terms leads to

$$(x^3 + 1/x^3) + a(x^2 + 1/x^2) + b(x + 1/x) + c = 0.$$

After introducing $u = x + 1/x$ we know

$$x^2 + 1/x^2 = (x + 1/x)^2 - 2 = u^2 - 2.$$

and

$$x^3 + 1/x^3 = (x + 1/x)^3 - 3(x + 1/x) = u^3 - 3u$$

so we obtain the cubic equation

$$u^3 - 7u^2 + 12u = 0.$$

This has roots of 0, 3, and 4. To find the roots of the original equation, we must now solve three equations:

$$\begin{aligned} x + 1/x &= 0 \\ x + 1/x &= 3 \\ x + 1/x &= 4, \end{aligned}$$

or, after rearrangement

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 - 3x + 1 &= 0 \\ x^2 - 4x + 1 &= 0. \end{aligned}$$

Therefore, the roots of the original equation are $\pm i$, $(3 \pm \sqrt{5})/2$, and $2 \pm \sqrt{3}$.

10. Show that if r is a non-zero root of an anti-palindromic polynomial so is $1/r$.

Solution:

If $r = 1$ then $1/r = r$ so in this case $1/r$ is a root if r is. So consider a root r different from 1. By virtue of the factorization $p(x) = (x-1)q(x)$ established in the prior problem, we see that r must be a root of the palindromic polynomial q. But that implies $1/r$ is also a root of q, and hence of p.