Administrivia

- Syllabus and other course information posted on the internet: links on blackboard & at www.dankalman.net.
- Check assignment sheet for reading assignments and exercises. Be sure to read about polynomials, including roots, factorization, long division and remainders.
- If you haven't done so already, get and install Freemat and start working on the tutorial.

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Matrinomials Lectures 1+2

Outline

- Basics: polynomials and roots
- Horner's Form and Quick Computation
- Products and Sums of Roots
- Reverse Polynomials
- Sums of Reciprocal Roots
- Long Division and Remainders
- Palindromials

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Reminder: What's a polynomial?

$$5x^3 - 7x^2 + 3x - 2$$

$$15x^2 - 14x + 3$$

$$5x^4 - 11x^3 + 6x^2 + 7x - 3$$

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Polynomials have ROOTS

$$5x^3 - 7x^2 + 3x - 1 = 0$$
$$x = 1$$

Roots are related to *Factors*

$$5x^3 - 7x^2 + 3x - 1 = (x-1)(5x^2 - 2x + 1)$$

Complete factorization …

$$5x^3 - 7x^2 + 3x - 1 = (x-1)\left(x - \frac{2+\sqrt{-16}}{10}\right)\left(x - \frac{2-\sqrt{-16}}{10}\right)$$

$$= (x-1)(x-.2-.4i)(x-.2+.4i)$$

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Complex Number Review

- Extension of the real numbers created by inventing a *nonreal* number $i = \sqrt{-1}$.
- Any combination of the form $a + bi$ with *a* and *b* real numbers is defined to be a complex number.
- These form a number system with addition, subtraction, and multiplication defined using the usual rules of algebra. Eg
  (3+7i)+(4-3i) = 7 + 4i;
  (3 + 7i)(4 – 3i) = 12 – 9i + 28i – 21i^2 (FOIL)
  = 12 + 19i + 21 (because i^2 = –1) = 33 + 19i.
- There is even division. Eg:
  $$\frac{3+7i}{4-3i} = \frac{3+7i}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{(3+7i)(4+3i)}{(4-3i)(4+3i)} = \frac{-9+37i}{25} = \frac{-9}{25} + \frac{37}{25}i.$$
- Historically, there has been a progression of different number systems developed as various types of numbers were recognized and defined: whole numbers, fractions (rational numbers), negative numbers, and irrational numbers (giving us the reals). The complex number system is just another step in this progression.

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Fundamental Theorem of Algebra

- Refers to any polynomial of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where the coefficients $a_n, a_{n-1}, \cdots, a_0$ are complex numbers and $a_n \neq 0$. This includes the case that the coefficients are real numbers.
- The theorem says that such a polynomial can always be expressed in this form:
  $$p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$
  where the roots $r_1, r_2, \cdots, r_n$ are complex numbers, not necessarily all distinct.
- This is an existence theorem: there must exist *n* complex roots (again not necessarily distinct). Finding them is another matter.

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Solutions by Radicals

- For quadratics, cubics, and quartics all roots expressible from coefficients using addition, subtraction, multiplication, division, and square-, cube-, and fourth-roots.
- Familiar quadratic formula: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Similar formula for roots of $x^3 + ax + b = 0$:

$$r = \sqrt[3]{\frac{-b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{\frac{-b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

- Even more complicated formula(s) for quartics.
- Abel-Galois-Ruffini Theorem: No such formulas exist for quintics and higher degree equations. For example, it can be shown that the quintic $3x^5 - 15x + 5$ has at least one root that is not expressible in terms of radicals. This proves that there is no general formula using radicals for all solutions of all quintics.

Wikipedia Excerpt

Interpretation [edit]

[from https://en.wikipedia.org/wiki/Abel–Ruffini_theorem]

The theorem does *not* assert that some higher-degree polynomial equations have *no* solution. In fact, the opposite is true: *every* non-constant polynomial equation in one unknown, with real or complex coefficients, has at least one complex number as a solution (and thus, by polynomial division, as many complex roots as its degree, counting repeated roots); this is the fundamental theorem of algebra. These solutions can be computed to any desired degree of accuracy using numerical methods such as the Newton–Raphson method or the Laguerre method, and in this way they are no different from solutions to polynomial equations of the second, third, or fourth degrees. It also does *not* assert that *no* higher-degree polynomial equations can be solved in radicals: the equation $x^n - 1 = 0$ can be solved in radicals for every positive integer n, for example. The theorem only shows that there is no *general solution in radicals* that applies to *all* equations of a given degree greater than 4.

The solution of any second-degree polynomial equation can be expressed in terms of its coefficients, using only addition, subtraction, multiplication, division, and square roots, in the familiar quadratic formula: the roots of the equation $ax^2 + bx + c = 0$ (with $a \neq 0$) are

$$\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Analogous formulas for third-degree equations and fourth-degree equations (using square roots and cube roots) have been known since the 16th century. What the Abel–Ruffini theorem says is that there is no similar formula for general equations of fifth degree or higher. In principle, it could be that the equations of the fifth degree could be split in several types and, for each one of these types, there could be some algebraic solution valid within that type. Or, as Ian Stewart wrote, "for all that Abel's methods could prove, every particular quintic equation might be soluble, with a special formula for each equation."[4] However, this is not so, but this impossibility is a strictly stronger result than the Abel–Ruffini theorem and is derived with Galois theory.

Finding Roots is Hard …

… But we *can* find out some things easily

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

- The sum of the roots is 11/5
- The average of the roots is 11/20
- The sum of the reciprocals of the roots is 7/3
- We'll get back to roots in a bit
  First let's look at computation
- Can you compute p(3) in your head?

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Horner's Form

- Standard descending form
  $$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$
- Horner form
  $$p(x) = (((5x - 11)x + 6)x + 7)x - 3$$
- Also referred to as partially factored or nested form

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Derivation of Horner Form

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$
$$= (5x^3 - 11x^2 + 6x + 7)x - 3$$
$$= ((5x^2 - 11x + 6)x + 7)x - 3$$
$$= (((5x - 11)x + 6)x + 7)x - 3$$

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Quick Evaluation

- Compute *p*(2):
  (((5·2 – 11)2 + 6)2 + 7)2 – 3
- Answer = 27
- Compute *p*(3):
  (((5·3 – 11)3 + 6)3 + 7)3 – 3
- Answer = 180
- Compute *p*(2/5):
  (((5·# – 11)# + 6)# + 7)# – 3
- Answer = 23/125?

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Getting Back to our Roots

Coefficients and combinations of roots

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Product of Roots

±Constant term / highest degree coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

(The ± sign is + because degree is even)
Product of the roots is …

– 3/5

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Key Idea of Proof

- For our example

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

- Say the roots are r, s, t, and u.
- $p(x) = 5(x - r)(x - s)(x - t)(x - u)$
- Multiply this out to find the constant term

  $5rstu = -3$

- Note constant term is 5(- r)(- s)(- t)(- u), so for odd degree we get an extra – sign

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Sum of Roots

– 2nd highest degree coefficient divided by highest degree coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Sum of the roots is …

11/5

Average of the roots is …

11/20

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A System To Remember

- Specified sum and product of two unknowns
- For example: this system $\begin{Bmatrix} x+y=5 \\ xy=8 \end{Bmatrix}$.
- Solution: x and y are the roots of the quadratic equation $t^2 - 5t + 8 = 0$.
- Reason: $(t-x)(t-y) = t^2 - 5t + 8$ iff $x + y = 5$ and $xy = 8$.
- The unknowns in any system of the form $\begin{Bmatrix} x+y=u \\ xy=v \end{Bmatrix}$ must be given by $\frac{u \pm \sqrt{u^2-4v}}{2}$.

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Exercises

1. Prove the sum of the roots result

2. Prove this: For any polynomial p , the average of the roots of p is equal to the average of the roots of the derivative p'.

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Reverse Polynomial

- Consider the polynomial

$$p(x) = x^6 - 7x^5 + 11x^4 + 13x^3 - 5x^2 + 2x + 1$$

- The reverse polynomial is

$$\text{Rev } p(x) = 1 - 7x + 11x^2 + 13x^3 - 5x^4 + 2x^5 + x^6$$

- Question: How are the roots of Rev *p* related to the roots of *p*?
- Answer: Roots of reverse polynomial are reciprocals of roots of the original.

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Sum of Reciprocal Roots

– 1st degree coefficient divided by the constant coefficient

Example:

$$p(x) = 5x^4 - 11x^3 + 6x^2 + 7x - 3$$

Sum of the reciprocal roots is …

7/3

Average of the reciprocal roots is …

7/12

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Exercises

1. Prove this: For any polynomial p of degree n with nonzero constant term,
   Rev $p(x) = x^n\, p(1/x)$.

2. Use 1 to prove that the roots of Rev $p(x)$ are the reciprocals of the roots of $p(x)$.

3. Use 2 to prove: if
   $p(x) = a_n x^n + \cdots + a_1 x + a_0$
   and $a_0 \neq 0$, then the sum of the reciprocals of the roots of p is $-a_1 / a_0$

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Polynomial Long Division

Example: $(x^2 - 5x + 6) \div (x-3)$

	x	-2	
x + -3)	x^2	-5 x +	6
	x^2	-3 x +	0
		-2 x +	6
		-2 x +	6
			0

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Polynomial Long Division

Redo the example working from the constants upward

$$
\begin{array}{r r l}
 & & -2 + x \\
\cline{3-3}
-3 + x & \big) & 6 - 5x + x^2 \\
 & & 6 - 2x + 0 \\
\cline{3-3}
 & & \quad -3x + x^2 \\
 & & \quad -3x + x^2 \\
\cline{3-3}
 & & \qquad\quad 0
\end{array}
$$

Polynomial Long Division

- Another example: $(x^2 – 5x + 6) \div (x-1)$
  (Do it on white board / scratch paper)
- Answer $-6 – x – 2x^2 – 2x^3 – 2x^4 – 2x^5 – \cdots$
- Similar to the long division $1 \div 3$ to find .333333…
- Alternate form of answer:
  $-6 – x – 2x^2 (1 + x + x^2 + x^3 \cdots)$
  $= -6 – x + 2x^2/(x-1)$
- Part in parentheses is a power series for the rational function $1/(1-x)$.
- Similar to mixed fraction form of answer to a division problem: fraction part = remainder / divisor

Amazing Application

- Start with $p(x) = x^3 - 2x^2 - x + 2$
- Find the derivative (Do it on whiteboard/scratch paper)
- Reverse both (Do it on whiteboard/scratch paper)
- Do a long division problem of the reversed p(x) into the reversed p'(x), working from the constants forward

(Do it on whiteboard/scratch paper)

- Answer: $3 + 2x + 6x^2 + 8x^3 + 18x^4\ldots$
- The coefficients have an astonishing interpretation: sums of powers of roots

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Checking the answer

- $p(x) = x^3 - 2x^2 - x + 2 = (x-2)(x^2 - 1)$
- Roots are 2, 1, and -1
- Sum of roots = 2
- Sum of squares of roots = 6
- Sum of cubes = 8
- Sum of fourth powers = 18
- Etc.

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Proof Hints

- Rev $p(x) = x^n\, p(1/x)$
- Logarithmic Derivative: $f\,'\,/\,f = (\ln f\,)'$
- If $f(x) = (x - r)\,(x - s)\,(x - t)\,\cdots$ then

$$(\ln f(x))' = \frac{1}{x-r} + \frac{1}{x-s} + \frac{1}{x-t} + \cdots$$

- Geometric Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{and} \quad \frac{1}{x-a} = \frac{-1/a}{1-x/a}$$

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Palindromials

- p(x) = reverse p(x)
- Recall: if p(r) = 0 ($r\neq0$) then [rev p](1/r) = 0.
- So for palindromials, whenever r is a nonzero root, so is 1/r.
- Example: $x^4 + 7x^3 - 2x^2 + 7x + 1$
- 1 and -1 are not roots, so roots come in reciprocal pairs
- Must factor as (x-r)(x-1/r)(x-s)(x-1/s)
- Rewrite: $(x^2 – ux + 1)\,(x^2 – vx + 1)$ where u = r+1/r and v = s + 1/s

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Matching Coefficients

- $(x^2 - ux + 1)(x^2 - vx + 1) = x^4 + 7x^3 - 2x^2 + 7x + 1$
- $u + v = -7$ and $uv + 2 = -2$
- Two unknowns. Sum = -7, product = -4
- They are the roots of $x^2 + 7x - 4 = 0$
- u and v are given by $\dfrac{-7 \pm \sqrt{65}}{2}$
- Our factorization is

$$\left(x^2 - \frac{-7+\sqrt{65}}{2}x + 1\right)\left(x^2 - \frac{-7-\sqrt{65}}{2}x + 1\right)$$

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Solve for x

$$\left(x^2 - \frac{-7+\sqrt{65}}{2}x + 1\right)\left(x^2 - \frac{-7-\sqrt{65}}{2}x + 1\right) = 0$$

- Use quadratic formula on each factor
- Roots from first factor are

$$\frac{1}{2}\left(\frac{-7+\sqrt{65}}{2} \pm \sqrt{\frac{98-14\sqrt{65}}{4}}\right) = \frac{1}{4}\left(-7+\sqrt{65} \pm \sqrt{98-14\sqrt{65}}\right)$$

- Remaining roots are

$$\frac{1}{4}\left(-7-\sqrt{65} \pm \sqrt{98-14\sqrt{65}}\right)$$

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General Reduction Method

- $p(x) = ax^6 + bx^5 + cx^4 + dx^3 + cx^2 + bx + a$
- $p(x)= x^3(ax^3 + bx^2 + cx + d + c/x + b/x^2 + a/x^3)$
- $p(x)/x^3 = a(x^3+1/x^3) + b(x^2+1/x^2) + c(x+1/x)+d$
- We want roots of $a(x^3+1/x^3) + b(x^2+1/x^2) + c(x+1/x)+d$
- Almost a polynomial in $u = (x+1/x)$.
- $u^2 = x^2 + 2 + 1/x^2 \rightarrow x^2+1/x^2 = u^2 – 2$
- $u^3 = x^3 + 3x + 3/x + 1/x^3 = x^3 + 3u + 1/x^3$
  $\rightarrow x^3+1/x^3 = u^3 – 3u$
- Leads to a cubic polynomial in u:
  $a(u^3 – 3u) + b(u^2 – 2) + c(u)+ d$
- Solve for u, and then substitute in $u = (x+1/x)$ and solve for x. Note $x^2 – ux + 1 = 0$.

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Example: $x^6 + 3x^5 + 2x^4 + 3x^3 + 2x^2 + 3x + 1$

- Let $p(x) = x^6 + 3x^5 + 2x^4 + 3x^3 + 2x^2 + 3x + 1$
- $p(x) = x^3(x^3 + 3x^2 + 2x + 3 + \frac{2}{x} + \frac{3}{x^2} + \frac{1}{x^3})$
- $\frac{p(x)}{x^3} = (x^3+\frac{1}{x^3}) + 3(x^2+\frac{1}{x^2}) + 2\left(x + \frac{1}{x}\right) + 3$
- Substitute $u = x + \frac{1}{x}$, using $x^2 + \frac{1}{x^2} = u^2 - 2$ and $x^3 + \frac{1}{x^3} = u^3 - 3u$
- $u^3 - 3u + 3u^2 - 6 + 2u + 3 = 0$
- $0 = u^3 + 3u^2 - u - 3 = (u+3)(u^2-1)$
- If $u = -3$ then $-3 = x + \frac{1}{x}$ so $x^2 + 3x + 1 = 0$, giving $x = \frac{-3\pm\sqrt{5}}{2}$.
- Roots obtained from $u = -1$ and $u = 1$ similarly.

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Example

$$x^8 + 3x^7 - 6x^6 + 12x^5 - 13x^4 + 12x^3 - 6x^2 + 3x + 1 = 0$$

- Make the standard reduction

$$u^4 + 3u^3 - 10u^2 + 3u + 1 = 0$$

- It's another palindromial! Reduce again

$$v^2 + 3v - 12 = 0$$

- Solve with quadratic formula $v = \dfrac{-3 \pm \sqrt{57}}{2}$
- Find u: $v = u + 1/u$ so $u^2 - vu + 1 = 0$
- Solve for u

$$u = \frac{-3 - \sqrt{57} \pm \sqrt{50 + 6\sqrt{57}}}{4}$$

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Solve for x

- We have found 4 values for u
- We know x + 1/x = u
- Solve $x^2 - ux + 1 = 0$ with quadratic formula for each known u value
- That gives 8 roots
- Here is one:

$$\frac{-3 - \sqrt{57} + \sqrt{50 + 6\sqrt{57}} + i\sqrt{(6 + 2\sqrt{57})\sqrt{50 + 6\sqrt{57}} - 52 - 12\sqrt{57}}}{8}$$

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