The Reduced Cubic

- In the prior example, we found the characteristic polynomial for a generic 3x3 circulant: $p(x) = (x-a)^3 - b^3 - c^3 - 3bc(x-a)$
- Simplification: let $z = x - a$, equivalently $x = z + a$.
- $p(z+a) = z^3 - 3bcz - b^3 - c^3 = f(z)$
- But how do we know the correct value of a?
- For each root z_i, we know $z_i + a = r_i$ is a root of our original p.
- Sum for all roots: $z_1 + z_2 + z_3 + 3a = r_1 + r_2 + r_3$.
- From earlier work, $r_1 + r_2 + r_3 = -(\text{quadratic coeff}) = 3$
- Similarly, $z_1 + z_2 + z_3 = -(\text{quadratic coeff in} f(z)) = 0$
- Therefore: $0 + 3a = 3$ so $a = 1$.
- In the original given polynomial
  $$p(z+1) = (z+1)^3 - 3(z+1)^2 + 4(z+1) - 12 = z^3 + z - 10$$
- This has no z^2 term, and is referred to as a *reduced* cubic

Degree *n* case

- We can *reduce* any monic polynomial similarly
- Say $g(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$
- Define $h(z) = g(z-b)$
- Then z_k is a root of $h(z)$ iff $t_k = z_k - b$ is a root of g.
- Define $b = \frac{a_{n-1}}{n}$. Then the sum of the roots of h is given by $\sum z_k = \sum(t_k + b) = (\sum t_k) + nb = -a_{n-1} + a_{n-1} = 0$.
- Conclusion: $h(z)$ has no z^{n-1} term.
- From the roots of h we can easily find the roots of g.
- When solving polynomial equations, we need only consider *reduced* polynomials.

Reduced Circulants

- Assume we are given a reduced polynomial $p(t)$
- We want to find a circulant matrix *C* having p for its characteristic polynomial
- The sum of the roots of p will be the sum of the eigenvalues of *C*. So the sum of the eigenvalues of *C* is zero.
- Theorem: in any square matrix, the sum of the eigenvalues equals the sum of the diagonal elements (called the *trace*)
- So the trace of *C* must equal 0.
- But all the diagonal elements of *C* are the same.
- Conclusion: the diagonal elements of *C* are all zero
- Recap: The characteristic polynomial of a circulant matrix *C* is reduced if and only if *C* has all zeros on its diagonal.
- When we try to find *C* given a reduced polynomial $p(t)$, we should assume the diagonal entries are all zero.

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Characteristic Polynomial

$$\det(xI - M) = \det \begin{bmatrix} x & -q_1 & -q_2 & \cdots & -q_{n-1} \\ -q_{n-1} & x & -q_1 & \cdots & -q_{n-2} \\ -q_{n-2} & -q_{n-1} & x & \cdots & -q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & -q_2 & -q_3 & \cdots & x \end{bmatrix}$$

CONCLUSION: General formula for the determinant of a circulant suffices.

Case $n = 3$

The characteristic polynomial of $M = q(W) = q_1 W + q_2 W^2$:

$$\begin{aligned} \det\left(xI - \begin{bmatrix} 0 & q_1 & q_2 \\ q_2 & 0 & q_1 \\ q_1 & q_2 & 0 \end{bmatrix}\right) &= \det\begin{bmatrix} x & -q_1 & -q_2 \\ -q_2 & x & -q_1 \\ -q_1 & -q_2 & x \end{bmatrix} \\ &= x^3 - q_1^3 - q_2^3 - 3q_1 q_2 x \\ &= x^3 - 3q_1 q_2 x - (q_1^3 + q_2^3) \end{aligned}$$

Solving the Cubic

- Given $p(x) = x^3 + ax + b$
- Characteristic polynomial of $q(W)$ is $x^3 - 3q_1 q_2 x - (q_1^3 + q_2^3)$
- Must solve the system

$$\begin{aligned} q_1^3 + q_2^3 &= -b \\ q_1 q_2 &= -a/3 \end{aligned}$$

- Roots of p will be $q(1) = q_1 + q_2$, $q(\omega) = q_1\omega + q_2\omega^2$, and

  $q(\omega^2) = q_1\omega^2 + q_2\omega$ where $\omega = \dfrac{-1 + i\sqrt{3}}{2}$

Finding q

$$\left\{ \begin{array}{rcl} q_1^3 + q_2^3 & = & -b \\ q_1 q_2 & = & -a/3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} q_1^3 + q_2^3 & = & -b \\ q_1^3 q_2^3 & = & -a^3/27 \end{array} \right\}$$

q_1^3 and q_2^3 are roots of $x^2 + bx - a^3/27 = 0$

Thus, q_1 and q_2 are given by

$$\left\{ \frac{-b \pm \sqrt{b^2 + 4a^3/27}}{2} \right\}^{1/3} \quad \text{or} \quad \left\{ \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \right\}^{1/3}$$

Solving the Quartic

- $x^4 + ax^2 + bx + c$
- Characteristic polynomial of $q(W)$

  $x^4 - (4q_3q_1 + 2q_2^2)x^2 - 4q_2(q_1^2 + q_3^2)x + (q_2^4 - q_1^4 - q_3^4 - 4q_1q_3q_2^2 + 2q_1^2q_3^2) = 0$
- Elimination leads to

$$q_2^6 + \frac{a}{2}q_2^4 + \left(\frac{a^2}{16} - \frac{c}{4}\right)q_2^2 - \frac{b^2}{64} = 0$$

- 4th roots of unity: $\pm 1, \pm i$
- Roots of p: $\pm q_1 + q_2 \pm q_3$, and $\pm i q_1 - q_2 \mp i q_3$.

Quintic and Beyond

- For any n, the nth roots of unity are expressible in terms of radicals
- Suppose p is a polynomials whose roots r_k are *not* expressible in terms of radicals (possible for degree 5 or more according to Galois Theory)
- If $q(W)$ is a circulant having p as characteristic polynomial, then each r_k is a linear combination of roots of unity and coefficients of q.
- Since the roots of unity are expressible in terms of radicals, the coefficients of q *cannot* be.
- Circulant Method Fails