These slides will be presented between slides 20 and 21 of the prior slide show (Solving polys with circulants parts 3 and 4).
Solving polys with circulants parts 3 and 4

Solving the Cubic

- Given $p(x) = x^3 + ax + b$ find $C = q(W) = q_1 W + q_2 W^2$
- Characteristic polynomial of $q(W)$ is $x^3 - 3q_1q_2x - (q_1^3 + q_2^3)$
- q_1 and q_2 $= \left\{ \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \right\}^{1/3}$
- Roots of p will be $q(1) = q_1 + q_2$, $q(\omega) = q_1\omega + q_2\omega^2$, and

  $q(\omega^2) = q_1\omega^2 + q_2\omega$ where $\omega = \frac{-1 + i\sqrt{3}}{2}$
- If $\frac{b^2}{4} + \frac{a^3}{27} < 0$ we need a cuberoot of a complex number. How do we find that?

- Example: what is $\sqrt[3]{\frac{365}{108} + i\frac{97}{24}}$?
- Convert to polar coordinates: $\frac{365}{108} + i\frac{97}{24} = r(\cos\theta + i\sin\theta)$
- One cube root will be $\sqrt[3]{r}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$
- $\sqrt[3]{r} = \sqrt[6]{\left(\frac{365}{108}\right)^2 + \left(\frac{97}{24}\right)^2}$, $\frac{\theta}{3} = \frac{1}{3}\tan^{-1}\frac{97/24}{365/108} = \frac{1}{3}\tan^{-1}\frac{873}{730}$
- Calculator: $\sqrt[3]{r} \cong 1.740051085$, $\frac{\theta}{3} \cong .2914567945$
- Calculator: $\sqrt[3]{r}\cos\frac{\theta}{3} \cong 1.666666667$, $\sqrt[3]{r}\sin\frac{\theta}{3} \cong .5000000000$
- This suggests $\sqrt[3]{\frac{365}{108} + i\frac{97}{24}} = \frac{5}{3} + i\frac{1}{2}$.
- In exact arithmetic, we can confirm that $\left(\frac{5}{3} + i\frac{1}{2}\right)^3 = \frac{365}{108} + i\frac{97}{24}$ proving $\sqrt[3]{\frac{365}{108} + i\frac{97}{24}} = \frac{5}{3} + i\frac{1}{2}$.

Solving Quartics with Circulants

- As with cubics, we consider only *reduced* quartic polynomials.
- For example: $p(x) = x^4 - 4x^2 + 8x + 35$.
- We want to find a 4x4 circulant matrix C (with 0 diagonal) having $p(x)$ for its characteristic polynomial.
- That means C will have the form $\begin{bmatrix} 0 & q_1 & q_2 & q_3 \\ q_3 & 0 & q_1 & q_2 \\ q_2 & q_3 & 0 & q_1 \\ q_1 & q_2 & q_3 & 0 \end{bmatrix}$
- And $C = q_1 W + q_2 W^2 + q_3 W^3$
- Characteristic polynomial is $\det \begin{bmatrix} x & -q_1 & -q_2 & -q_3 \\ -q_3 & x & -q_1 & -q_2 \\ -q_2 & -q_3 & x & -q_1 \\ -q_1 & -q_2 & -q_3 & x \end{bmatrix}$

WolframAlpha computational knowledge engine.

det({{x,-q_1,-q_2,-q_3},{-q_3,x,-q_1,-q_2},{-q_2,-q_3,x,-q_1},{-q_1,-q_2,-q_3,x}})

Web Apps Examples Random

Input interpretation:

$$\begin{vmatrix} x & -q_1 & -q_2 & -q_3 \\ -q_3 & x & -q_1 & -q_2 \\ -q_2 & -q_3 & x & -q_1 \\ -q_1 & -q_2 & -q_3 & x \end{vmatrix}$$

Result: $-2 q_2^2 x^2 - 4 q_1 q_3 x^2 - 4 q_2 q_3^2 x - 4 q_1^2 q_2 x - q_1^4 + q_2^4 - q_3^4 + 2 q_1^2 q_3^2 - 4 q_1 q_2^2 q_3 + x^4$

$$x^4 - (4q_3q_1 + 2q_2^2)x^2 - 4q_2(q_1^2 + q_3^2)x + (q_2^4 - q_1^4 - q_3^4 - 4q_1q_3q_2^2 + 2q_1^2q_3^2)$$

Want this to equal $p(x) = x^4 - 4x^2 + 8x + 35$.

$$x^4-(4q_3q_1+2q_2^2)x^2-4q_2(q_1^2+q_3^2)x+(q_2^4-q_1^4-q_3^4-4q_1q_3q_2^2+2q_1^2q_3^2)$$

Want this to equal $p(x) = x^4 - 4x^2 + 8x + 35.$

Solve this system for the q's:

$$4q_3q_1 + 2q_2^2 = 4$$
$$-4q_2(q_1^2 + q_3^2) = 8$$
$$q_2^4 - q_1^4 - q_3^4 + 2q_1^2q_3^2 - 4q_1q_3q_2^2 = 35$$

Rearranged system:

$$4q_3q_1 + 2q_2^2 = 4$$
$$-4q_2(q_1^2 + q_3^2) = 8$$
$$q_2^4 - (q_1^2 + q_3^2)^2 + 4(q_1q_3)^2 - 4q_1q_3q_2^2 = 35$$

Rearranged system:

$$4q_3q_1 + 2q_2^2 = 4$$
$$-4q_2(q_1^2 + q_3^2) = 8$$
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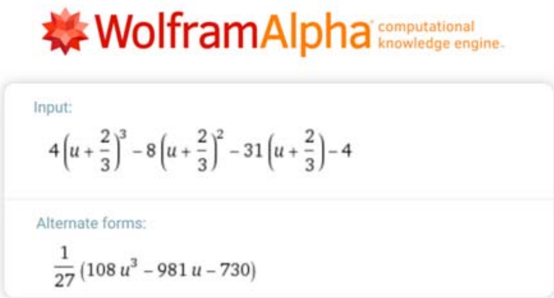
$$2q_3q_1 = 2 - q_2^2$$
$$(q_1^2 + q_3^2) = -\frac{2}{q_2}$$
$$q_2^4 - \frac{4}{q_2^2} + (2 - q_2^2)^2 - 2(2 - q_2^2)q_2^2 = 35$$

$$q_2^6 - 4 + q_2^2(q_2^4 - 4q_2^2 + 4) - 2q_2^4(2 - q_2^2) = 35q_2^2$$

$$4q_2^6 - 8q_2^4 - 31q_2^2 - 4 = 0$$

Let $z = q^2$: $\quad 4z^3 - 8z^2 - 31z - 4 = 0$

- $4z^3 - 8z^2 - 31z - 4 = 0$
- To reduce, factor: $4\left(z^3 - 2z^2 - \frac{31}{4}z - 1\right)$
- Let $z = u + \frac{2}{3}$ in $4z^3 - 8z^2 - 31z - 4 = 0$



- Monic: $u^3 - \frac{109}{12}u - \frac{365}{54} = 0$

- $u^3 - \frac{109}{12}u - \frac{365}{54} = u^3 + au + b$
- Discriminant:

$$\frac{1}{4}b^2 + \frac{1}{27}a^3 = \left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3 = -\left(\frac{97}{24}\right)^2$$

$$= \left(\frac{365}{108}\right)^2 - \left(\frac{109}{36}\right)^3$$

$$= \left(\frac{365}{3\cdot 36}\right)^2 - \left(\frac{109}{36}\right)^3$$

$$= \frac{36^3\cdot 365^2 - 9\cdot 36^2\cdot 109^3}{108^2\cdot 36^3}$$

$$= \frac{36\cdot 365^2 - 9\cdot 109^3}{108^2\cdot 36}$$

$$= \frac{4\cdot 365^2 - 109^3}{36^2\cdot 36} = \frac{-81\cdot 97^2}{36^2\cdot 36}$$

- $u = \sqrt[3]{\frac{365}{108} + i\frac{97}{24}} + \sqrt[3]{\frac{365}{108} - i\frac{97}{24}} = \frac{5}{3} + i\frac{1}{2} + \frac{5}{3} - i\frac{1}{2} = \frac{10}{3}$
- $z = u + \frac{2}{3} = 4$
- $q_2^2 = z = 4 \implies q_2 = \pm 2$
- We don't have to find all possible solutions – just one
- Let us consider the case $q_2 = 2$.
- $2q_3q_1 = 2 - q_2^2 = -2$ so $q_3q_1 = -1$ and $q_1^2q_3^2 = 1$.
- $(q_1^2 + q_3^2) = -\frac{2}{q_2} = -1$
- Again we have a sum/product system, so q_1^2 and q_3^2 must be the roots of the quadratic $t^2 + t + 1$. ie $q_1^2, q_2^2 = \frac{-1 \pm i\sqrt{3}}{2}$

- $q_1^2, q_2^2 = \frac{-1 \pm i\sqrt{3}}{2}$
- These are the nonreal cuberoots of unity, and so expressible as $\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}$.
- If we take $q_1^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ then we can halve the angle to find $q_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.
- Then we have to take $q_3 = \frac{-1}{q_1}$
- Since q_1 is a sixth root of unity, $\frac{1}{q_1} = \overline{q_1}$. Thus $q_3 = -\overline{q_1} = -(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

- We have found
  $q_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $q_2 = 2$ and $q_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
- If these are correct, they give us a circulant for which the characteristic polynomial is our original quartic.
- The eigenvalues are:

$$q_1 + q_2 + q_3 = 2 + i\sqrt{3}, \quad -q_1 + q_2 - q_3 = 2 - i\sqrt{3}$$
$$iq_1 - q_2 - iq_3 = -2 + i, \quad -iq_1 - q_2 + iq_3 = -2 - i$$

- These are in fact the roots of our original quartic, which factors as
  $p(x) = (x^2 - 4x + 7)(x^2 + 4x + 5)$

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