Formal Problem Set 1

Due 2/2/2018 in Class

Format and style guidelines: see this webpage:
http://www.dankalman.net/AUhome/classes/classesS18/matrinomials/day1/mathwriting.html.

Collaboration Rules: It is not permitted to give or receive a complete solution to/from another student. It is not permitted to search for a solution to any problem on the internet or other reference. It is not permitted to copy a solution from any source. You may discuss the ideas of a problem with other students or with the instructor. Whether or not you do so, for every problem, you are required to compose your own solution in your own words.

1. Prove this: If $p(x)$ is any polynomial of degree at least 2, the average of the roots of $p(x)$ is equal to the average of the roots of $p'(x)$, the derivative of $p(x)$.

2. Find one exact real root of the polynomial $p(x) = x^6 - 2x^5 - 2x + 1$.

3. Let $p(x) = x^5 + 3x^4 - 3x^3 + 2x^2 - x - 12$. Find the sum of the cubes of the roots of p. That is, if the exact roots are represented by r_1, r_2, r_3, r_4, and r_5, find the exact value of $r_1^3 + r_2^3 + r_3^3 + r_4^3 + r_5^3$.

4. A polynomial $p(x)$ is called *anti-palindromic* if Rev $p(x) = -p(x)$. Prove that $p(x)$ is anti-palindromic if and only if $p(x) = (x\text{-}1)q(x)$ where $q(x)$ is palindromic.

5. The Fibonacci numbers are defined as follows. The first two Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, and for every k greater than 1, F_k is the sum of the two preceding Fibonacci numbers. In equation form, $F_k = F_{k-1} + F_{k-2}$, for $k > 1$. The first several Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, … .

   Prove: For every natural number k, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^k = \begin{bmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{bmatrix}$. [Hint: use an induction proof.]

Optional Additional Problems: The following problems seem to me to be more difficult than those above, though difficulty is in the eye of the beholder, so to speak. These are optional for students seeking an additional challenge.

6. Let $p(x)$ be a polynomial of degree n, with roots $r_1, r_2, \cdots, r_n$ (not necessarily distinct). Prove that

$$\frac{\text{Rev } p'(x)}{\text{Rev } p(x)} = \sum_{k=0}^{\infty} s_k x^k$$

   where $s_k = r_1^k + r_2^k + \cdots + r_n^k$. This was posed as problem 6 on the first problem set, where several hints were provided. In your proof, do not worry about convergence of the power series above.

7. The polynomial $p(x) = x^5 - 1$ has five complex roots. One root is evidently 1. Find the remaining four roots as exact algebraic expressions. [**Note:** you may have seen a method for solving this equation in another class or elsewhere, possibly involving sines and cosines. If so, please approach the problem afresh, using methods from our class. Do *not* go back and consult those prior sources. If you have already seen a solution using our methods, and remember what that solution is, do not submit a solution to this problem.]

8. Let $\alpha = \sqrt[3]{10 + 6\sqrt{3}}$ and $\beta = \sqrt[3]{10 - 6\sqrt{3}}$. Using a calculator to compute values of these constants, it appears that $\alpha + \beta = 2$. However, that is not a conclusive result – it is possible that $\alpha + \beta$ is actually just a very very close approximation to 2. Follow the outline below to prove that $\alpha + \beta$ does in fact equal 2 exactly.

    a. Show that $\alpha + \beta$ is a root of a cubic polynomial $p(x)$ by cubing $\alpha + \beta$ and relating the result to lower powers of $\alpha + \beta$.

    b. Use methods of calculus to demonstrate that your polynomial $p(x)$ has exactly one real root.

    c. By direct calculation, show that $p(2) = 0$.

    d. Explain why these steps combine to show that $\alpha + \beta = 2$.