Formal Problem Set 2

Due 2/23/2018 in Class

Format and style guidelines: see this webpage:
http://www.dankalman.net/AUhome/classes/classesS18/matrinomials/day1/mathwriting.html.

Collaboration Rules: It is not permitted to give or receive a complete solution to/from another student. It is not permitted to search for a solution to any problem on the internet or other reference. It is not permitted to copy a solution from any source. You may discuss the ideas of a problem with other students or with the instructor. Whether or not you do so, for every problem, you are required to compose your own solution in your own words.

1. The following equation is given: $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$.

   Using the equation, express $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k$ as a single 2×2 matrix with entries that depend on k.

2. Assume that A is a diagonalizable matrix, with characteristic polynomial $p(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$. Do not assume that the eigenvalues are distinct. Prove that $p(A)$ is equal to the zero matrix.

3. For the matrix A shown at right, find all the eigenvalues and their corresponding eigenspaces. Is A diagonalizable? Justify your answer.

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

4. Find the eigenvalues for the 4×4 W matrix; for each eigenvalue find one nonzero eigenvector. [Recall that W is obtained from the identity matrix by shifting the top row to the bottom.]

5. Let X be the matrix shown at right. Find an equation for X^k and prove your equation is correct.

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Optional Additional Problems: The following problems seem to me to be more difficult than those above, though difficulty is in the eye of the beholder, so to speak. These are optional for students seeking an additional challenge.

6. Let A be an $n \times n$ matrix with entry a_{ij} in row i and column j. We know that the characteristic polynomial is

$$p(\mathrm{t}) = \det(tI - A) = t^n + ct^{n-1} + (\text{lower order terms})$$

   for some coefficient c. Prove that $c = -(a_{11} + a_{22} + \cdots + a_{nn})$. [Note: this shows that for any square matrix, the sum of the diagonal elements is equal to the sum of the eigenvalues.]

7. Suppose that a matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. For each j let S_j be a linearly independent set of (one or more) eigenvectors for λ_j. Prove that the union $S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent set.

8. Let $p(t) = t^3 + at + b$. Find a 3×3 circulant matrix C with zero diagonal (that is, one of the form $\begin{bmatrix} 0 & u & v \\ v & 0 & u \\ u & v & 0 \end{bmatrix}$) whose characteristic polynomial is equal to $p(t)$. Then use this information to find one root of $p(t)$ (which will be an eigenvalue of C).