Formal Problem Set 3

Due 4/20/2018 in Class

Format and style guidelines: see this webpage:
http://www.dankalman.net/AUhome/classes/classesS18/matrinomials/day1/mathwriting.html.

Collaboration Rules: It is not permitted to give or receive a complete solution to or from another student. It is not permitted to search for a solution to any problem on the internet or other reference. It is not permitted to copy a solution from any source. You may discuss the ideas of a problem with other students or with the instructor. Whether or not you do so, for every problem, you are required to compose your own solution in your own words.

1. **[Note: This problem refers to *circulant*, not *companion*, matrices]** Let P be the eigenvector matrix we considered for $n \times n$ circulant matrices. That is,

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix},$$

where $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$ is the n^{th} root of unity in the first quadrant of the complex plane that has the smallest positive polar coordinates angle θ.

Also, let $\mathcal{D}_n$ be the set of $n \times n$ diagonal matrices and $\mathcal{C}_n$ be the set of $n \times n$ circulant matrices. Note that these are both vector spaces: their elements can be added, multiplied by scalars, and the rules of vector algebra hold for these operations.

Finally, define a function $T(A) = P^{-1}AP$ where A can be any $n \times n$ matrix, and P is the matrix defined above. As we learned in class, for any circulant matrix C, $P^{-1}CP$ is a diagonal matrix. Therefore, the function T maps $\mathcal{C}_n$ into $\mathcal{D}_n$. In symbols, $T: \mathcal{C}_n \to \mathcal{D}_n$.

   a. Prove that T is a linear transformation from $\mathcal{C}_n$ into $\mathcal{D}_n$. That is, prove that for any circulant matrices C_1 and C_2, and for any scalar s, the following identities hold:
      (i) $T(C_1 + C_2) = T(C_1) + T(C_2)$
      (ii) $T(sC_1) = sT(C_1)$.

   b. Prove that T is a one to one function (also known as an injection). Then explain why the null space of T must consist of only the zero matrix.

   c. Prove that T is an *onto* function (also known as a surjection). Hint: Note that $\mathcal{C}_n$ and $\mathcal{D}_n$ are both n dimensional – a basis for $\mathcal{C}_n$ is provided by the matrices $I, W, W^2, \cdots, W^{n-1}$ and a basis for $\mathcal{D}_n$ is provided by the n matrices that have a single diagonal entry equal to one, and all other entries equal to zero. Using these facts, argue that the range of T must actually be all of $\mathcal{D}_n$.

   d. According to part c, there must be a 4×4 circulant matrix C for which $T(C) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

      Find the matrix C.

2. Let C be any $n \times n$ companion matrix. **[Note: *C* is a *companion*, not a *circulant*, matrix.]**

   a. Prove that the matrices $I, C, C^2, \cdots, C^{n-1}$ are linearly independent. That means, that no nontrivial linear combination of these matrices can equal the zero matrix. Hint: Compute the first rows of these matrices. What do these first rows imply about the first row of a linear combination of the matrices?

   b. For any polynomial $f(t)$ of degree $n-1$ or less, prove that $f(C)$ cannot equal the zero matrix.

3. A sequence $\{a_k\}_{k=0}^{\infty}$ is defined by the conditions $a_0 = 0,\ a_1 = 0,\ a_2 = 1$, and for all $k \ge 0$, $a_{k+3} = a_{k+2} + 8a_{k+1} + 6a_k$. Find an equation for a_k as a function of k, and verify (with a calculator, if you wish) that it gives the correct values for a_3 and a_4. Hint: The characteristic polynomial has a root you can find by guessing.

4. A sequence $\{a_k\}_{k=0}^{\infty}$ is defined by the conditions $a_0 = 1,\ a_1 = -4,\ a_2 = 6$, and for all $k \ge 0$, $a_{k+3} = a_{k+2} + 8a_{k+1} + 6a_k$. Note: this is not the same sequence mentioned in problem 3, but it *does* satisfy the same difference equation. Complete the following steps for this sequence. In all parts of this problem the value of n is 3.

   a. Find the matrix $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & E_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & E_n & \cdots & E_{2n-4} & E_{2n-3} \\ 1 & E_n & E_{n+1} & \cdots & E_{2n-3} & E_{2n-2} \end{bmatrix}$ from equation 2 of the lecture outline. Use the difference equation to find the necessary values of E_n, E_{n+1}, etc..

   b. Find the matrix $\begin{bmatrix} -c_1 & -c_2 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$ from equation 2 of the lecture outline.

   c. Multiply the matrices from steps a. and b. together and verify that they are inverses of each other.

   d. Compute the coefficient vector $\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & 1 \\ -c_2 & -c_3 & -c_4 & \cdots & 1 & 0 \\ -c_3 & -c_4 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-3} \\ a_{n-2} \\ a_{n-1} \end{bmatrix}$ for the given difference equation and initial terms.

   e. Verify that $a_k = b_0E_k + b_1E_{k+1} + b_2E_{k+2} + \cdots + b_{n-1}E_{k+n-1}$ holds for $k = 0, 1, \ldots, n-1$.

f. Explain why these steps imply that the sequence defined by

$$\{a_k\} = b_0\{E_k\} + b_1 L\{E_k\} + b_2 L^2\{E_k\} + \cdots + b_{n-1}L^{n-1}\{E_k\}$$

is the solution to the given difference equation with the specified initial terms.

g. Find the polynomial $f(t) = \sum_{m=0}^{n-1} b_m t^m$ discussed in the lecture outline paragraph 3e.

h. Express a_k as a function of *k* in the form $\sum_{j=1}^{n} \frac{f(\lambda_j)}{p'(\lambda_j)} \lambda_j^k$ where $p(t)$ is the characteristic polynomial for the given difference equation and $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, are the roots of $p(t)$.

Optional Additional Problems: The following problems seem to me to be more difficult than those above, though difficulty is in the eye of the beholder, so to speak. These are optional for students seeking an additional challenge.

5. For a given square matrix A, a *minimal polynomial* is a monic $m(t)$ polynomial of minimal degree for which $p(A) = 0$ (the zero matrix).
   a. The set of $n \times n$ matrices is a vector space of dimension n^2. Use this fact to prove that every such matrix has a minimal polynomial of degree at most n^2.
   b. Suppose that $f(t)$ is *any* polynomial for which $f(A) = 0$, and suppose $m(t)$ is a minimal polynomial for A. Prove that $m(t)$ divides evenly into $f(t)$ (*ie*. with no remainder).
   c. Use part b. to show that the minimal polynomial for any square matrix A is unique.

6. A Pascal's triangle matrix of size n is an $n \times n$ triangular matrix with 1's on the main diagonal, 1's in the first column, and with the nonzero entries of row i equal to the entries of the corresponding row of Pascal's triangle. For example, the 4×4 Pascal's triangle matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Find and prove a formula for the inverse of the $n \times n$ Pascal's triangle matrix.