# Application of Polynomial Division with Remainders

Sometimes we need to substitute a root from one polynomial into another polynomial. In such a case, the algebra can be simplified significantly using polynomial division. This is illustrated below.

**Problem:** find local maxima and minima for the function

$$f(x) = 4x^5 - 5x^4 - 20x^3 + 20x^2 + 40x - 11.$$

**Solution:** We compute $f'(x) = 20x^4 - 20x^3 - 60x^2 + 40x + 40$. Inspired by the 20's and 40's we rewrite this as $f'(x) = 20x^4 - 20x^3 - 20x^2 - 40x^2 + 40x + 40$, and so obtain the factorization $f'(x) = 20(x^2 - 2)(x^2 - x - 1)$. This shows that $f' = 0$ when $x = \pm\sqrt{2}$, and when $x = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2}(1 \pm \sqrt{5})$. These are the critical values for $f(x)$.

Next we have to compute $f(x)$ at each critical value. For the purpose of this example we will only be concerned with doing this for one of the critical values, namely $\alpha = \frac{1}{2}(1 + \sqrt{5})$. If we substitute this directly into the equation for $f(x)$, we can carry out a lot of algebra and eventually find a simple exact expression for the result. Polynomial long division provides a short cut.

We know that $\alpha$ is a root of the polynomial $p(x) = x^2 - x - 1$. If we divide $p(x)$ into $f(x)$ and find the quotient $q(x)$ and the remainder $r(x)$, that means

$$f(x) = p(x)q(x) + r(x),$$

so

$$f(\alpha) = p(\alpha)q(\alpha) + r(\alpha) = 0 \cdot q(\alpha) + r(\alpha) = r(\alpha).$$

But the remainder has to have lower degree than the divisor, $p(x)$, so $r(x)$ is linear and computing $r(\alpha)$ will be much easier than computing $f(\alpha)$ directly.

On the next page, the details of the long division process are shown. Postponing that for a minute, let us turn directly to the resulting remainder: $r(x) = 25x - 9$. Substituting, we find

$$r(\alpha) = 25\alpha - 9 = 25 \cdot \frac{1}{2}\left(1 + \sqrt{5}\right) - 9 = \frac{25}{2} + \frac{25}{2}\sqrt{5} - 9 = \frac{7}{2} + \frac{25}{2}\sqrt{5}.$$

That is the exact value of $f(\alpha)$.

To complete the example, the long division below verifies that the remainder is given by $r(x) = 25x - 9$.

$$
\begin{array}{r|rrrrrr}
 & 4x^3 & -\;x^2 & -\;17x & +\;2 & & \\ \hline
x^2 - x - 1 & 4x^5 & -\;5x^4 & -\;20x^3 & +\;20x^2 & +\;40x & -\;11 \\
 & 4x^5 & -\;4x^4 & -\;4x^3 & & & \\ \cline{2-4}
 & & -x^4 & -\;16x^3 & +\;20x^2 & & \\
 & & -x^4 & +\;x^3 & +\;x^2 & & \\ \cline{3-5}
 & & & -17x^3 & +\;19x^2 & +\;40x & \\
 & & & -17x^3 & +\;17x^2 & +\;17x & \\ \cline{4-6}
 & & & & 2x^2 & +\;23x & -\;11 \\
 & & & & 2x^2 & -\;2x & -\;2 \\ \cline{5-7}
 & & & & & 25x & -\;9
\end{array}
$$