Elementary Mathematical Models and a Glimpse of Chaos

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Today’s slides: www.dankalman.net
EMM resources:

www.dankalman.net/emm
Development Goals

- Intrinsic value/interest/significance
- Coherent story line
- Show power of algebra in context
- General education perspective on how math actually gets applied
- Decreased emphasis on abstract manipulative skills
- Highlight college algebra topics most likely to appear in client discipline introductory courses
Persistent Themes

• Discrete Sequential Data: $a_1, a_2, a_3, \cdots$ and approximating models; applying math through models

• Recursive patterns are easy to formulate: dependence of $a_{n+1}$ on $a_n$. Difference equations

• Solutions to difference equations: explicit equation for $a_n$ as a function $f(n)$; extension to continuous models

• Parameterized families of difference equations and solutions; fitting a model to actual data by choosing best values for parameters

• Direct prediction: evaluate $f(n)$ to predict data value number $n$

• Inverse Prediction: invert $f(n)$ to predict for which $n$ the data value will reach a specified value

• Graphical, Numerical, and Theoretical methods
Arithmetic Growth

- Each term increases by fixed amount over preceding term
- Example: population grows by 5000 each year
- Difference equation $p_{n+1} = p_n + 5000$
- Solution: $p_n = p_0 + 5000n$
- Typical questions: What will the population be in year $n$? When will population reach 60000?
- In general: $a_{n+1} = a_n + d; \ a_n = a_0 + dn$
- Lead in to topic of linear equations
Quadratic Growth

- Computer network example. Adding one computer to a network of $n$ requires adding $n$ additional communications lines.

- Difference equation $c_{n+1} = c_n + n$

- Solution: $c_n = n(n - 1)/2$

- Typical questions: How many lines are needed for 200 computers? We have 500 lines, how many computers can we put on the network?

- In general: Each term increases over preceding term by an amount that depends linearly on $n$.

- In general: $a_{n+1} = a_n + d + en$; $a_n = a_0 + dn + en(n - 1)/2$

- Constant second differences

- Sums of arithmetic growth models

- Lead in to topic of quadratic functions and equations
**Functional Equation Example**

Difference equation: \( a_{n+1} = a_n + 22 + 10n \) with \( a_0 = 5 \)

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\[
a_n = 5 + 22n + 10(1 + 2 + \cdots + n) = 5 + 22n + 10n(n + 1)/2
\]
Geometric Growth

- Each term is a fixed multiple of the preceding term; equivalently, each term increases by a constant percentage over the preceding term.

- Example: population doubles each year (increases by 100%).

- Difference equation $p_{n+1} = 2p_n$.

- Solution: $p_n = p_02^n$.

- Typical questions: What will the population be in year $n$? When will population reach 60000?

- In general: $a_{n+1} = ra_n; a_n = a_0r^n$.

- Lead in to topic of exponential functions and logs.
Mixed Arithmetic and Geometric Growth

- Each term combines a fixed multiple of the preceding term with a fixed increment;

- Example: Pollution flows out of a lake in proportion to the existing concentration, but flows into the lake at a constant absolute rate

- Difference equation $p_{n+1} = .9p_n + 3$ (one tenth of the pollution flows out, and three more units are added, each unit of time)

- Solution: $p_n = p_0(.9^n) + 3(1 - .9^n)/(1 - .9)$

- Typical questions: What will the pollution load be in year $n$? When will it reach 100?

- In general: $a_{n+1} = ra_n + d; \ a_n = a_0 r^n + d(1 - r^n)/(1 - r)$

- Realistic Applications: repeated drug doses, repeated loan payments or investments
Logistic Growth

• Modified version of geometric growth. Each term is a multiple of the preceding term, but the multiplier varies linearly with the size of the term.

• Example: population $p$ goes up in a year by a factor of $0.01(200 - p)$.

• Difference equation $p_{n+1} = 0.01(200 - p_n)p_n$.

• No explicit solution, but interesting qualitative behavior: initial growth similar to exponential, but levels off.

• In general: $a_{n+1} = m(L - a_n)a_n$.

• Interesting analysis results: $0 < mL < 1 \Rightarrow a_n \rightarrow 0$; $1 \leq mL < 3 \Rightarrow a_n \rightarrow L - 1/m$; $0 \leq mL < 4 \Rightarrow a_n \in [0, L] \forall n$;
Fixed Points

• Condition: \(a_{n+1} = a_n\)

• Logistic Growth: \(a_{n+1} = [m(L - a_n)]a_n\)

• Need \(m(L - a_n) = 1\)

• Fixed point = \(L - 1/m\)
Harvesting

• Diff Eqn: \( a_{n+1} = m(L - a_n)a_n - h \)

• Fixed Point equation

\[
\begin{align*}
m(L - a_n)a_n + h &= a_n \\
m(L - x)x + h &= x \\
x^2 + (1 - mL)x - h &= 0
\end{align*}
\]

• Generally two theoretical fixed points

• Fixed points key to analysis
Chaos in Logistic Growth Model

Logistic growth model can always be transformed to the quadratic family iteration by change of variable. This is a classical example in the study of chaos. In the context of real population model it is possible to introduce and explore concepts like:

- nonlinearity of the difference equation
- stability of the model under perturbations of parameters
- periodic orbits
- sensitive dependence on initial conditions
- bifurcation diagrams
- chaos