An Extension of the Remainder Theorem

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1. The Coefficient Formulas

If $p(x)$ is a polynomial and c is a constant, the division algorithm for polynomials asserts the existence of a polynomial $q(x)$ satisfying

$$p(x) = (x - c)q(x) + R.$$

The remainder theorem shows that the remainder R is a constant and numerically equal to $p(c)$. The following theorem produces corresponding formulas for the coefficients of the quotient, $q(x)$.

THEOREM. *Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *and define* $p_j(x)$ *by* $p_j(x) = a_n x^n + \cdots + a_j x^j$ *for* $j = 0,1,2,\ldots,n$. *Then the coefficient* b_k *of* x^k *in the quotient* $q(x)$ *is given by* $b_k = \dfrac{p_{k+1}(c)}{c^{k+1}}$.

NOTE: For $k = -1$, the right side of the formula is simply $p(c)$, the remainder.

Proof. From the remainder theorem $p(x) = (x-c)q(x) + p(c)$; hence $p(x) - p(c)$ is divisible by $x - c$ with quotient $q(x)$. The coefficients of q may be obtained by performing the division $\dfrac{p(x) - p(c)}{x - c}$:

$$\frac{p(x) - p(c)}{x - c} = \frac{\sum_{j=0}^{n} a_j(x^j - c^j)}{x - c} = \sum_{j=1}^{n} a_j\left(\frac{x^j - c^j}{x - c}\right)$$

(note that the term for $j = 0$ drops out of the sum). Now applying the identity $x^j - c^j = (x - c)\sum_{k=0}^{j-1} x^k c^{j-k-1}$ yields:

$$(1)\qquad \frac{p(x) - p(c)}{x - c} = \sum_{j=1}^{n} a_j \sum_{k=0}^{j-1} x^k c^{j-k-1}.$$

To find the coefficient of each power of x, the order of summation is interchanged:

$$\frac{p(x) - p(c)}{x - c} = \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} a_j x^k c^{j-k-1} = \sum_{k=0}^{n-1} x^k \left(\sum_{j=k+1}^{n} a_j c^{j-(k+1)} \right)$$

$$= \sum_{k=0}^{n-1} x^k \left(\sum_{j=k+1}^{n} \frac{a_j c^j}{c^{k+1}} \right) = \sum_{k=0}^{n-1} x^k \frac{p_{k-1}(c)}{c^{k-1}}.$$

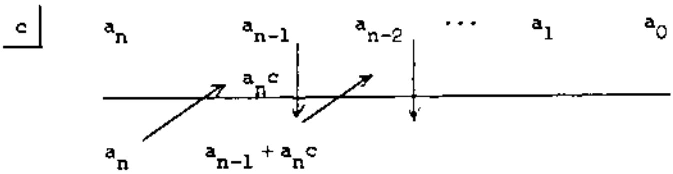
This verifies the formula for the coefficients of q and complet the proof.

2. Synthetic Division

It is frequently convenient to express a polynomial p in the partially factored (or "nested") form

$$p(x) = ((\cdots(((a_n x + a_{n-1})x + a_{n-2})x + a_{n-3})\cdots + a_2)x + a_1)x + a_0.$$

This form is quite useful both for computer applications and for human computation, since it allows calculation of the value of $p(x)$ as a running operation without recalling intermediate values. To determine $p(c)$, begin with the leading coefficient; multiply by c and add the next coefficient; multiply by c again and add the next coefficient; and so on. Coincidentally, the result after each round of multiplying and adding is one of the coefficients of q. For example, at the first stage there is only a_n which is also b_{n-1}. After multiplying and adding once, the result is $a_n c + a_{n-1}$, or b_{n-2}. Each successive iteration yields the corresponding coefficient of q: after n cycles, the value of $p(c)$ is obtained. These remarks inspire a computational format for calculating the value of $p(c)$ that will yield the coefficients of q as well. At each stage, the result after multiplying by c and adding the appropriate coefficient of p must be tabulated, multiplied by c, and added to another coefficient in turn.



The entries below the line are the coefficients of q. After recording one of these entries, its product with c is entered above the line in the next column to the right (suggested by the diagonal arrows). The numbers in this column are added to produce the next entry below the line (suggested by the vertical arrows). The final value below the line is $p(c)$.

The computational format described above is the familiar form of synthetic division. It is presented here as a method for computing $p(c)$ that also gives the quotient q. It is interesting that this method of division can be derived by a method so far removed from the usual one: that of streamlining the long division algorithm. The derivation above also emphasizes the intimate connection between evaluating a polynomial at c and dividing it by $x - c$.

3. Differentiation

In the previous sections, the identity

$$\frac{p(x) - p(c)}{x - c} = q(x)$$

has been exploited to derive the coefficients of q. Another application arises from the observation that the left side of the identity is a difference quotient for p. The derivative of p may thus be evaluated by finding the limit of $q(x)$ as c approaches x. The notation $q_c(x)$ will be employed in the following discussion to represent explicitly the dependence of q on c. For each value of c, dividing $p(x) - p(c)$ by $x - c$ produces a particular quotient $q_c(x)$.

The difference quotient for p, $q_c(x)$, is a polynomial in c. Hence the derivative of p—that is, $\lim_{c \to x} q_c(x)$—may be determined by replacing c by x in the formula for q. From equation (1) we have

$$\frac{p(x) - p(c)}{x - c} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} a_j x^k c^{j-1-k};$$

hence

$$\lim_{c \to x} \frac{p(x) - p(c)}{x - c} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} a_j x^k x^{j-1-k} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} a_j x^{j-1}.$$

Since the expression $a_j x^{j-1}$ does not depend upon k, the summation on k may be carried out, giving the following formula for $p'(x)$:

$$p'(x) = \sum_{j=1}^{n} j a_j x^{j-1}.$$

Careful analysis of the quotient q has produced a simple derivation of the general rule for differentiation of a polynomial.

Two common approaches to differentiating polynomials are derivation of the power rule from the binomial theorem and a combination of the product rule with induction. The derivation given above provides an elegant alternative. To differentiate x^n, proceed as follows:

$$\frac{d}{dx}(x^n) = \lim_{c \to x} \frac{x^n - c^n}{x - c} = \lim_{c \to x} x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1} = nx^{n-1}.$$