

Pythagorean n -ples

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Pythagorean Triples

Equivalent Defs: Integer triple (a, b, c) for which

- $a^2 + b^2 = c^2$
- $\left(\frac{a}{c}, \frac{b}{c}\right)$ is rational pt on unit circle
- (a, b) is integer vector with integer length c .

Parameterization

- Take $u > v \in \mathbb{Z}^+$
 $(u^2 - v^2, 2uv, u^2 + v^2)$ is always a PT
- Example: $(u, v) = (3, 2) \Rightarrow (a, b, c) = (5, 12, 13)$

Pythagorean 4-tuples (P4Ts)

Equivalent Defs: Integer 4tuple (a, b, c, d) for which

- $a^2 + b^2 + c^2 = d^2$
- $(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ is rational pt on unit sphere
- (a, b, c) is integer vector with integer length d .

Parameterization

- Params $u, v, w \in \mathbb{Z}$ give rise to P4T

$$(|u^2 - v^2 - w^2|, 2uv, 2uw, u^2 + v^2 + w^2)$$

- Example:

$$(u, v, w) = (3, 1, 2) \Rightarrow (a, b, c, d) = (4, 6, 12, 14)$$

$$16 + 36 + 144 = 196 = 14^2$$

Compare Parameterizations

- PT: $(u^2 - v^2, 2uv, u^2 + v^2)$
- P4T: $(|u^2 - v^2 - w^2|, 2uv, 2uw, u^2 + v^2 + w^2)$
- P5T:
 $(|u^2 - v^2 - w^2 - x^2|, 2uv, 2uw, 2ux, u^2 + v^2 + w^2 + x^2)$
- PnT: (a_1, a_2, \dots, a_n) where
 $u_1, u_2, \dots, u_{n-1} \in \mathbb{N}$, and
 $a_1 = |u_1^2 - \sum_{k=2}^{n-1} u_k^2|$
 $a_k = 2u_1 u_k$ for $k = 2 \dots n - 1$
 $a_n = u_1^2 + \sum_{k=2}^{n-1} u_k^2$

Derivation

- Inspired by Manjul Bhargava's 2011 Hedricks Lecture derivation for PTs
- Characterizes rational points on unit circle as intersections of the circle with lines
- If a line has rational slope, and passes through one rational point on the circle, the other intersection will also be a rational point
- This argument extends perfectly to intersections of lines with the unit sphere in n dimensions

Three Dimensional Case

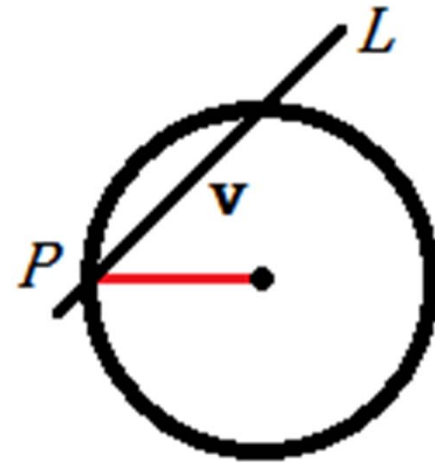
- (a, b, c, d) is a P4T iff $\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$ is a rational point on unit sphere in \mathbb{R}^3 .
- Consider a line through $(-1, 0, 0)$ parallel to a rational vector, and hence to an integer vector (u, v, w)
- Assume $u \neq 0$ so the line is not tangent to the sphere
- We find the other point of intersection, show it is a rational point

Key Equivalence

- Let L be a line through $P = (-1,0,0)$ not tangent to the sphere.
- There is a point $Q \neq P$ where L meets the sphere
- Claim: Q is a rational point iff L is parallel to a rational vector (iff parallel to an integer vector)
- (\Rightarrow) Assume Q is a rational point. Know P is a rational point. $\therefore Q - P$ is rational and $L \parallel Q - P$
- (\Leftarrow) Assume $L \parallel \mathbf{v}$ for some integer vector \mathbf{v} .
Then $L = \{P + t\mathbf{v} \mid t \in \mathbb{R}\}$. Next slide: Intersect with sphere to find Q , show it is a rational point.

Solving for Q

- Line L : all points $\mathbf{r}(t) = P + t\mathbf{v}$
- L not tangent to sphere so $P \cdot \mathbf{v} \neq 0$
- \mathbf{r} on unit sphere iff $\mathbf{r} \cdot \mathbf{r} = 1$
- $(P + t\mathbf{v}) \cdot (P + t\mathbf{v}) = 1$
- $P \cdot P + 2tP \cdot \mathbf{v} + t^2\mathbf{v} \cdot \mathbf{v} = 1$
- But $P = (-1, 0, 0) \Rightarrow P \cdot P = 1$
- $2tP \cdot \mathbf{v} + t^2\mathbf{v} \cdot \mathbf{v} = 0$
- $t(2P \cdot \mathbf{v} + t\mathbf{v} \cdot \mathbf{v}) = 0$
- $t = 0$ or $t = \frac{-2P \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \equiv t_1$
- $P = \mathbf{r}(0)$; $Q = \mathbf{r}(t_1)$



Finding Rational Points

- Line L through P parallel to the integer vector $\mathbf{v} = (u, v, w)$
- L not tangent to the sphere at P so $u \neq 0$.
- $Q = \mathbf{r}(t_1) = P + t_1 \mathbf{v} = (-1, 0, 0) + t_1(u, v, w)$
- $Q = (-1, 0, 0) + \frac{-2P \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} (u, v, w)$
 $= (-1, 0, 0) + \frac{2u}{\mathbf{v} \cdot \mathbf{v}} (u, v, w)$
- $Q = \frac{1}{\mathbf{v} \cdot \mathbf{v}} \{(-\mathbf{v} \cdot \mathbf{v}, 0, 0) + (2u^2, 2uv, 2uw)\}$
- $Q = \frac{1}{u^2 + v^2 + w^2} (u^2 - v^2 - w^2, 2uv, 2uw)$
- $(u^2 - v^2 - w^2, 2uv, 2uw, u^2 + v^2 + w^2)$ is P4T

Example

- $\mathbf{v} = (u, v, w) = (1, 2, 3)$
- P4T is $(12, 4, 6, 14)$
- Divide out 2 to find a *primitive* P4T $(6, 2, 3, 7)$
- We also obtain all the integer multiples
- Also can permute first three entries
 $(6, 3, 2, 7)$, $(2, 6, 3, 7)$, $(6, 2, 3, 7)$, etc
- Is there an efficient way to choose (u, v, w) 's so that we obtain every possible P4T without any duplication?

As some of you may know ...

Bhargava's derivation of $(u^2 - v^2, 2uv, u^2 + v^2)$ was presented in a 2011 Math Horizons article coauthored with Nathan Carter. It parodied the then popular Harry Potter books/movies. We called it *Harvey Plotter and the Circle of Irrationality*.

Today's presentation cries out for a sequel

Harvey Plotter 2

Fourples of Power*

