Linear Algebra

Computer Project 2: Linear Transformation Geometry

Overview and Instructions

For this project you should work with a partner. If you wish you can work in a small group, but it should have at most four members. Be sure that each member of your group gets a chance to work the computer controls. As explained later, your team will be assigned to produce a lab writeup.

As discussed in class, matrix-vector multiplication is used to define a special kind of function which operates on vectors and produces vector results. In this project you will explore the geometric nature of some of these functions for the special case of 2×2 matrices, which define functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, meaning that they operate on vectors in $\mathbb{R}^2$ and produce as results vectors in $\mathbb{R}^2$. We will look at graphs in which both the vector that is operated on, and the result, are represented by points in the xy plane.

Here is a very simple example, using the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Applying this matrix to a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ produces the result $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Graphically, the vector that is operated on is considered to be a point (x, y) and the result is the point $(2x, 2y)$. Using the function notation from calculus, we have a function defined by $f(x, y) = (2x, 2y)$.

Although you can not make a graph of this function in the same manner used in calculus, there are ways to visualize the operation of the function as a *mapping*, that is, a way to transport each point in the plane to another point in the plane. In this lab you will explore how this works using a computer activity hosted on a webpage. A link to the webpage and instructions for operating it are provided on a separate handout. If you have not yet done so, look at that now to get a feel for how the webpage works.

Lab Instructions. In each section below, a set of matrices is described that share similar geometric characteristics. For each type, experiment with several examples and formulate a general description of your observations.

As you go along, a record should be kept for the group showing your steps and conclusions. This can be either on paper or in an MS Word document. If you want to use Word, open a document now and save it on your G drive or on a memory stick. The advantage of using Word is that you can cut and paste images from the webpage into your lab record, as explained in the other handout.

Every member of your team should be given a copy of the lab record. It will be needed in a follow-up assignment based on today's work. More details are given at the end of the lab instructions.

Scalar Matrices

Scalar Matrices have the form $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$, where r can be any real number. For example, choose $r = 2$. Enter the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ in the webpage. Use the different drawing options to experiment with this mapping, as illustrated in the other handout. For example use the rectangle tool to draw letter shapes. Or use curves, lines, and circles to make some simple but recognizable image, like a face. Based on your observations, how would you describe the effect of this matrix?

Now write a section about your conclusion in your lab record. State what matrix you used and what you observed. If you are using MS Word, you can include an image showing your matrix A and the graph windows.

Next, repeat the process using a different number for r, and again report the results in your lab record.

Repeat this entire process a few more times. At the end you should have several different values of r, including positive and negative values, and also values both greater than 1 and less than 1.

Finally, write a concluding statement about diagonal matrices in your lab record. It should describe in general what the effect of a scalar matrix is, allowing for all possible values of r.

Diagonal Matrices

Diagonal Matrices have the form $\begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, where r and s can be any two real numbers. You have already seen what happens when they are the same number. In this section, consider what happens when they are different numbers. One varation to consider is to choose r and s with opposite signs. Try with at least 2 different choices for the r and s values. For each example you try, follow the same procedures as you did with scalar matrices. That is, choose a specific example, experiment with it in the webpage until you understand what the transformation does and then enter that information (both what matrix you tried and what results you observed) in your lab record. Then repeat for one or more different examples of a diagonal matrix. Finally, write a conclusion section about diagonal matrices.

Summary Instructions for Rest of Lab. For the remaining matrix types follow the same procedures as for scalar and diagonal matrices. In each case experiment with a few different examples, entering the matrices and your observations in your lab record. Conclude each of these sections of the lab record by including a summary statement for whatever matrix type you have been studying.

Cosine-Sine Matrices

Cosine-Sine Matrices have the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ can be any angle. Use at least three different values of θ. You can do this by computing the sine and cosine of an angle on the calculator. A convenient alternative is to use special triangles like the 3-4-5 triangle. For the 3-4-5 triangle, the cosine and sine are 3/5 and 4/5. Some other special triangles are 5-12-13, 7-24-25, and 8-15-17. If you use these to create examples, also figure out the angles involved. For example, if you use the 3-4-5 triangle with $\cos\theta = 3/5$ and $\sin\theta = 4/5$, then $\tan\theta = 4/3$ so you can find θ using the $\tan^{-1}$ button on a calculator.

You should discover a simple description for how a Cosine-Sine matrix transforms a geometric pattern. How does the angle θ relate to your description?

ab Matrices

The *ab* Matrices have the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, for any two real numbers a and b. Try at least two different sets of values for a and b. These matrices are closely related to the cosine-sine matrices. Can you see how, both geometrially and algebraically?

Shear Matrices

Shear Matrices have the form $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$ where r is any real number. Try some of each form, with both positive and negative values of r.

Cosine-Sine Matrices - second form

These matrices have the form $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, where θ can be any angle. The simplest cases are found with θ given by 90 or 180 degrees. You can make up other examples in the same manner used for the earlier cosine-sine matrices. As before, try to determine how the angle θ is related to the geometric behavior you observe.

Projection Matrices

Projection Matrices have the form $\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ where $a^2 + b^2 = 1$. The simplest examples have a or b equal to 0 or ± 1. Another simple example is to define both a and b to equal $\cos \pi/4 = \sqrt{2}/2$. For some other simple examples, use the special triangles (like 3-4-5) and let $a = \cos\theta$ and $b = \sin\theta$. Using the 3-4-5, $a = 3/5$ and $b = 4/5$, for a matrix $\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$.

Lab Writeup Prepare a writeup from the lab. It should be a neatly organized presentation of the sections described above, telling for each type of matrix what you observed, and what conclusions you drew about that type of matrix. If you have followed the instructions about putting information in your lab record, it is probably fine as is for the lab report. If your lab record is an MSWord document, you should proof read it and correct any errors or expand explanations where necessary before printing a final copy. For handwritten lab records, if your record is messy or disorganized, or if you want to make editorial improvements, you should do a rewrite.

If you worked with a partner or a group, there will be one writeup for which you will all share credit. One copy will be turned in for my review. It should have the names of all team members on it. After it is returned to you, each team member should make a copy of the graded report for inclusion in his or her course portfolio. The completed reports will be due the Monday following our exam. However, you may want to review your findings before the due date to help prepare for the exam.