Another Way to Sum a Series: Generating Functions, Euler, and the Dilog Function
Historical Appendix

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This is a supplement to [10]. The main paper recounts a demonstration by Lewin [11] that $\sum 1/k^2 = \pi^2/6$ and asks incidentally whether this demonstration might have been known to Euler. Here we look into the question in greater detail than was possible within the scope of the original paper. Although we have no definitive conclusions to report, readers interested in the history of Euler's work may find some value in retracing our investigations. Many original sources are now readily available via the internet, and we provide live links to those we cite.

Lewin's Argument. Consider the function

$$\mathrm{Li}_2(z) = \sum \frac{1}{k^2} z^k = z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots, \tag{1}$$

called by some the *dilog* function. As shown by Euler [5],

$$\mathrm{Li}_2(-1/z) + \mathrm{Li}_2(-z) + \frac{1}{2}(\ln z)^2 = C \tag{2}$$

where C is a constant. This can be verified by showing that the derivative of the left-hand side of (2) is zero.

Taking $z = 1$ in (2) leads to

$$C = 2\mathrm{Li}_2(-1) = 2(-1 + 1/4 - 1/9 + 1/16 - \cdots). \tag{3}$$

This can be related to $\mathrm{Li}_2(1)$ with a well known trick. The even terms of $\sum 1/k^2$ have sum $E = \sum 1/(2k)^2 = (1/4)\mathrm{Li}_2(1)$. Therefore the odd terms must sum to $D = (3/4)\mathrm{Li}_2(1)$, and the alternating sum in (3) is $E - D = -(1/2)\mathrm{Li}_2(1)$. This shows that $C = -\mathrm{Li}_2(1)$. Hence, (2) becomes

$$\mathrm{Li}_2(-1/z) + \mathrm{Li}_2(-z) + \frac{1}{2}(\ln z)^2 = -\mathrm{Li}_2(1). \tag{4}$$

Next substituting $z = -1$, we find

$$2\mathrm{Li}_2(1) + \frac{1}{2}[\ln(-1)]^2 = -\mathrm{Li}_2(1)$$

so that

$$\mathrm{Li}_2(1) = -\frac{1}{6}[\ln(-1)]^2.$$

To complete the analysis, we recall another of Euler's identities:

$$e^{i\pi} = -1$$

and so

$$i\pi = \ln(-1).$$

This tells us that $[\ln(-1)]^2 = (i\pi)^2 = -\pi^2$, and thus

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}.$$

The Historical Question. Did Euler know this proof? The key identities are Euler's, so he certainly had all of the necessary ingredients. In addition the style of analysis is similar to Euler's style.

Before we proceed a few comments are in order. First, though Lewin [11, p. 4] is our source for the demonstration above, we are not certain that it originated with him.

He gives a reference for Euler's key identity (2), and mentions earlier work by Euler on Li_2, but does not credit Euler for the specific argument evaluating $\sum 1/k^2$. Neither does he claim the argument as his own, saying only that the result is well-known but the derivation "is perhaps not so familiar." When Lewin wrote those words, maybe the argument was part of the folklore among specialists concerned with dilog and its brethren. If so, Lewin himself might not have known where the argument originated. Beyond seeking evidence of Lewin's argument in Euler's papers, we have not attempted to determine Lewin's source.

Second, in citing Euler's publications, we provide references (and links) to the most readily available editions of his works – i.e. the scanned images available online at the Euler Archive [3], as well as the *Opera Omnia*, the modern reprinting of his collected work. For each publication, we also provide the Eneström index number, a valuable aid regardless of which source one uses. In those cases where it is necessary to cite a specific passage in one of Euler's works, paragraph or section numbers are used, as these are fairly consistent across different printings and translations.

In retracing the history of Euler's work, it is important to distinguish between a date of publication (which is included in the bibliographical citations) and the times when his discoveries were made. In some cases the publications themselves indicate a date of presentation before a learned society, frequently far in advance of the publication date. The Eneström index also specifies dates of completion for some works, and Euler's correspondence provides another means for dating his discoveries. In considering whether Euler was aware of Lewin's argument, we will see that timing may be significant. Where we specify dates of particular results, they are generally as reported in [1, 2, 14].

What did he know and when did he know it? Euler studied the function that we now call dilog as early as 1730, when he discovered the identity

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) = C. \tag{5}$$

Note that this was before his first derivation of the $\pi^2/6$ result. In fact, he used (5) to give his first estimate of $\sum 1/k^2$, correct to 6 decimal places [4].

We know that Euler and Daniel Bernoulli were investigating the dilog power series for in April 1742, Euler wrote [9, p. 491ff] to Daniel Bernoulli, commending him on his derivation of $1+\frac{1}{4}+\frac{1}{9}+\cdots=\int_0^1 \ln\left(\frac{1}{1-x}\right)\frac{dx}{x}$ using "derivatives and integrals of series... in a much more useful and ingenious way than I." Euler notes that he too was only able to obtain that same result, nothing further. By August of 1742, Euler had discovered the value of $\mathrm{Li}_2(1/2)$ and was corresponding [8, p. 144ff] with Christian Goldbach on the dilogarithm function as an object of study.

According to Lewin [11], Euler also discussed dilog in his three volume treatise on integral calculus, the *Institutionum calculi integralis* [6], although Lewin does not specify where dilog appears. We believe Lewin is referring to paragraphs 196 – 200 (in Chapter 4) of the first volume originally published in 1768. Maximon's article [12] on the dilogarithm notes that [6] is widely cited as the first study of the dilogarithm function; he too finds just this one passage on the dilog there.

So what is in those paragraphs? Integration by parts is applied to $\int \frac{\ln x}{1-x}dx$ in order to derive (5), leading toward the discovery of several specific numerical values of the dilogarithm function. Along the way, Euler evaluates the constant C in (5) using the fact that $\mathrm{Li}_2(1)=\pi^2/6$. This suggests that at this time Euler considered the $\pi^2/6$ result to be settled fact, requiring no further substantiation.

Finally, in 1779, at the age of 72, Euler presented (2) in a paper dedicated to the dilog function [5]. This work includes a great many dilog identities. Although the methods are similar to those mentioned in [6], here they were used to carry the analysis of dilog much further than in any other paper we have discovered. In particular, this is the first instance we have found of the critical identity (2). In this work, too, Euler continued to treat $\mathrm{Li}_2(1)=\pi^2/6$ as well established fact, using it to evaluate constants of integration. He apparently saw no need to provide a derivation.

Our limited efforts to determine whether Euler ever published an argument like Lewin's thus produced no smoking gun. On the other hand, this is *Euler*. Is it conceivable that, with all of the necessary identities and methods at his finger tips, he failed to notice Lewin's argument? We think not. Either we failed to find where he wrote about it, or possibly he knew the argument but declined to write about it. This is, in itself, an intriguing possibility. If it is true, what were Euler's motivations?

In this regard timing seems to be vitally important. For example, when did Euler first discover (2)? As Sandifer [14] has explained, Euler was interested in derivations of the $\pi^2/6$ result for an extended period. His first proofs in 1735 used methods that drew some criticism. Over the next decade he continued to refine and develop these methods, deriving known results with them as one form of validation. But in 1741 he provided an additional derivation, this time using only elementary tools, Taylor series and integration by parts. After that, he no doubt considered the result to be beyond question. Consequently, if he discovered (2) (and along with it the Lewin argument) much later than 1741, there would have been little motivation for publishing an additional evaluation of $\mathrm{Li}_2(1)$. In particular, this would make sense if his first discovery of (2) was in the 1779 paper.

It is interesting to compare the methods he used to derive dilog identities in 1730 to those in the later works of 1768 and 1779. Varadarajan [15] has described how, in the earliest paper, (5) is used to estimate $\sum 1/k^2$ to six decimal places. In a private communication [16] he gives details about Euler's derivation of (5). Using Varadarajan's notation, Euler considered

$$T(\alpha) = \frac{1}{\alpha} + \frac{1}{2(\alpha+1)} + \frac{1}{3(\alpha+2)} + \cdots,$$

which reduces to $\sum 1/k^2$ when $\alpha = 1$. Then Euler obtained for $0 < u < 1$

$$T(\alpha) = \sum_{r=0}^{\infty} \frac{u^{r+\alpha}}{(r+\alpha)(r+1)} + \sum_{r=0}^{\infty} \left(\begin{array}{c} (r+1-\alpha) \\ r \end{array} \right) \frac{(1-u)^{r+1}}{(r+1)^2}$$
$$- \log(1-u) \sum_{r=0}^{\infty} \left(\begin{array}{c} (r+1-\alpha) \\ r \end{array} \right) \frac{(1-u)^{r+1}}{(r+1)}.$$

Taking $\alpha = 1$ produces (5).

In contrast, the 1768 and 1779 papers use methods we would consider quite elementary today. For example, the derivation of (2) in [5] proceeds as follows. Let $p = \int \frac{\ln y}{x}\, dx$

and $q = \int \frac{\ln x}{y}\, dy$. Then

$$p + q = \ln x \cdot \ln y + C \tag{6}$$

by what is essentially integration by parts.

Now supposing that $y = x - 1$, Euler says

$$\ln y = \ln(x-1) = \ln x + \ln(1 - \frac{1}{x})$$

and using the series expansion for $\ln(1+t)$ leads to

$$\ln y = \ln x - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} - \cdots.$$

Therefore

$$\begin{aligned} p &= \int \frac{\ln y}{x}\, dx \\ &= \frac{(\ln x)^2}{2} + \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \cdots \\ &= \frac{(\ln x)^2}{2} + \mathrm{Li}_2\left(\frac{1}{x}\right). \end{aligned}$$

Similarly, because $x = y + 1$,

$$\ln x = \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} + \cdots$$

so

$$\begin{aligned} q &= \int \frac{\ln x}{y}\, dy \\ &= \frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} + \cdots \\ &= -\mathrm{Li}_2(-y). \end{aligned}$$

Substituting the expressions for p and q in (6) then yields

$$\frac{(\ln x)^2}{2} + \mathrm{Li}_2\left(\frac{1}{x}\right) - \mathrm{Li}_2(1-x) = \ln x \cdot \ln(x-1) + C,$$

or after rearrangement,

$$\mathrm{Li}_2\left(\frac{1}{x}\right) - \mathrm{Li}_2(1-x) = \ln x \cdot \ln\left(\frac{x-1}{\sqrt{x}}\right) + C.$$

To modern eyes at least, this is much simpler than the earlier work. Even without knowing Latin, it's easy to follow Euler's steps in the original paper. Moreover, this process is a virtual identity machine: by positing various relationships between x and y, Euler produces a raft of dilog identities, including (2). This suggests (2) was not discovered by Euler in his earliest work with dilog and might have been unknown prior to the 1779 paper.

Taking a different tack, Roy [13] says perhaps Euler knew the Lewin argument but was reticent to publish because evaluating $\ln(-1)$ as $i\pi$ would be controversial. Again, timing is significant. As early as 1728, Euler used a result of Johann Bernoulli's to argue against Bernoulli's stated view that $\ln(-1) = 0$. Johann Bernoulli had shown in 1702 that the area of a segment of a circle of radius a with sine y and cosine x is given by $\frac{a^2}{4\sqrt{-1}} \ln \frac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$. Euler noted that in the case of the first quadrant, where $x = 0$, this formula reduces to $\frac{a^2}{4\sqrt{-1}} \ln(-1)$. Since the area of the quadrant is finite, it follows that $\ln(-1)$ cannot be zero. (See [1]; in particular, had Euler taken the additional step of comparing this actual area of the quadrant with $\frac{a^2}{4\sqrt{-1}} \ln(-1)$, he would have deduced that $\ln(-1) = \pi\sqrt{-1}$.)

Bradley [1] argues that Euler arrived at a complete understanding of the complex logarithmic function between 1743 and 1746: in correspondence with D'Alembert in 1746, Euler gave the identity $\ln(-a) = \ln(a) + \pi(1 \pm 2n)\sqrt{-1}$ [1]. D'Alembert raised a host of objections, and after a series of letters eventually the matter was dropped. A few months later Euler presented to the Berlin Academy a paper on logarithms of negative and imaginary quantities [7]. Writing to D'Alembert in August 1747, Euler claimed to have fully resolved the question of $\ln(-a)$ in his paper [7], "...where I believe I have put this matter to rest; at least for my part, I have not the least difficulty with it, whereas I had previously been extremely perplexed" (quoted in [1]).

Apparently, at the time of Euler's definitive evaluation of $\sum 1/k^2$, he still harbored some uncertainty about the meaning of $\ln(-1)$. This may be evidence in favor of Roy's idea. If Euler was aware of (2) prior to 1746, his confidence in Lewin's argument would likely have been undermined by uncertainties about the logarithms of negative quantities. If his discovery of (2) came later than 1746, his interest in Lewin's argument would likely have been diluted by the feeling that $\sum 1/k^2$ was well established.

We may never know whether Euler was aware of Lewin's argument. As we said earlier, given Euler's amazing creativity and insight, once he had (2), it seems to us unlikely that he would not have thought of Lewin's argument. Our historical investigations lend weight to this position, suggesting that his discovery of (2) either came too early (and so while he was still uncertain about $\ln(-1)$) or too late (and so after he had provided an airtight evaluation of $\sum 1/k^2$). But these speculations are not altogether convincing. We hope that further research in Euler's papers and correspondence may throw additional light on this question.

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