Sample Worksheets

On the following pages are several worksheets to reinforce ideas discussed in the main body of this paper. A brief summary and some comments about usage appear below. The following short-hand terminology will be used. Courses with a traditional unit on continuous logistic functions, modified as described in the main body of the paper, will be referred to as the *continuous setting*. Similarly, the *discrete setting* refers to courses with a unit on the traditional logistic growth difference equation, again modified as described in the main body.

All of the worksheets direct students to use excel spreadsheets that have been posted on-line. For illustrative purposes, the data used in the worksheet calculations have been pre-entered in the posted spreadsheets. Instructors who wish to use different data sets have two options: (1) They can preload their data values into the spreadsheet and save the resulting modified version for distribution to their students; or (2) they can explain to the students how to enter a new data set into the spreadsheet. Further details are included in the directions that appear in the upper left-hand corner of each spreadsheet.

Worksheet 1. This worksheet can be used in either the continuous or discrete settings. The intent is to give students hands-on experience with the refined logistic growth difference equation, and with the solution expressed as a continuous logistic function. The difference equation and subscript notation are de-emphasized so that students in the continuous setting need not deal with an unfamiliar formalism. Instructors in the discrete setting may wish to modify the worksheet to make use of their students' familiarity with difference equations.

Note 1. At step 3, home screen iteration on a graphing calculator will reduce the effort of calculating the table entries, and provide for greater accuracy. Here is how that would work on a TI calculator:

Step Number	Key In This	How Input Will Be Displayed	How Output Will Be Displayed
1	10 [enter]	**10**	**10**
2	÷ (.0025 [Ans] + .5) [enter]	**Ans/(.0025Ans + .5)**	**19.0476**
3	[enter]		**34.7826**
4	[enter]		**59.2953**

Note that Ans refers to the calculator's stored value from its prior computation, keyed in as [2nd][(–)], and that pressing ÷ in step 2 automatically inserts Ans as the start of the computation. Each successive press of the enter key then repeats the computation of step 2.

Note 2. Step 6 of the worksheet directs students to use page 1 of an xls file available on the internet. Page 4 of the same xls file can be also be used to verify the results of step 4, as follows. Copy the position numbers from page 1 and paste them (using *paste values*) into the *x* column on

page 4. Similarly copy the terms (green column) from page 1 into the y column on page 4. Then enter the parameters $A = 200$, $B = 19$, and $R = 0.5$ on page 4.

Worksheet 2. This worksheet guides students to formulate a refined logistic growth difference equation from two hypothetical experiments. The goal is to reinforce the students' understanding of the defining concepts of refined logistic growth. This is probably most appropriate in the discrete setting. Some instructors in that setting may wish to combine the first two worksheets into a single activity.

Worksheet 3. On this worksheet students will fit a continuous logistic growth function to a set of data. This can be used in both the continuous and discrete settings. My initial values of A, B, and r were 8, 79, and 0.9886 based on a y intercept of 0.1 and an inflection point of (380,4). Trial and error refined the parameters to 7.8, 210, and 0.986 with an average absolute error of 0.122. The excel optimizer found parameter values of 7.8765, 356.5, and .98456 with an average absolute error of 0.096200038.

To carry out trial and error refinement of parameters the students are directed to an excel spreadsheet posted on-line. It has a tab labeled *Find A, B, R* that computes the three parameters from entered values for the y-intercept (y_T) and the coordinates (x_F, y_F) of the inflection point. Although not discussed in the worksheet instructions, this tab can be useful if students wish to experiment with changes to y_T and (x_F, y_F) rather than A, B, and R.

Worksheet 4. On this worksheet students will fit a refined logistic growth difference equation to a set of data. It will make the most sense in the discrete setting. Instructors in that setting may wish to have students complete this activity first, and then revisit the same model in a continuous setting using worksheet 3.

When I completed the worksheet, I started with coefficients $m = 0.00007$ and $b = .62$ from the *find m and b* tab, and with $p_0 = 267$ that produced a mean absolute error of 131.9 on the *diff eqn + data points* tab. By trial and error I refined that to $p_0 = 304, m = 0.00007$, and $b = 0.63$ producing an error of 81.1. Excel's optimizer found a minimal error of 79.9 with the following parameters: $p_0 = 316.2078997$, $m = 6.93383 \cdot 10^{-5}$, and $b = 0.633351536$.

Refined Logistic Population Model Worksheet 1

A refined discrete logistic growth model produces a sequence of numerical values using the following rule: *If our current value is p, the next value will be given by r(p)·p where r(p) is the reciprocal of a linear function.* In this worksheet you will explore one specific example of such a model. It begins with a starting p value of 10 and will use the equation $r(p) = \frac{1}{0.0025p+0.5}$.

1. For the initial value $p = 10$, compute the factor $r(10)$ and the next p value.

$r(10) =$ _____   next p = _____

2. To compute this more easily on a calculator, we can represent the computation as
$$p \to \frac{p}{0.0025p + 0.5}.$$
Starting with $p = 10$, compute the next p using this approach, and verify that you get the same result as in step 1.
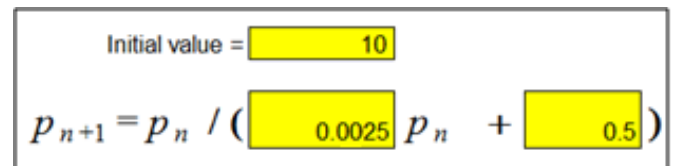
3. Using the same process as in step 2, complete the following table. Assume the first data value is for time 0, the second for time 1, and so on. Round your entries to 4 decimal places.

time (t)	population value (p)	extra column for step 4.
0	10	
1	19.0476	
2		
3		
4		
5		
6		
7		

4. For each t in column 1 of the table, compute $f(t) = \frac{200}{1+19\cdot0.5^t}$ and enter the result in the last column of the table. Verify that this produces the same p values as you found in step 3.

5. Use algebra to show that the function $f(t)$ from step 4 can be written as $f(t) = \frac{200}{1+19 \cdot e^{-kt}}$ where $k = \ln 2 \approx 0.693147$.

6. The calculation of population values in step 3 can be expressed succinctly in the equation $p_{n+1} = p_n / (0.0025 p_n + 0.5)$. The *diffeqn* tab of the excel file Primus-Logistic-Models.xlsx available at http://www.dankalman.net/logistic/primus/ can be used to obtain a table and a graph of the p values. The figure above illustrates how to enter the equation and starting population. Use the excel file to check your table entries above, and to see the shape of the graph. Describe below what this model predicts for the future growth of the population.

Initial value = [10]

$$p_{n+1} = p_n / (\boxed{0.0025} \; p_n + \boxed{0.5})$$

Refined Logistic Population Model Worksheet 2

A laboratory is developing a procedure to grow a certain kind of mold that will be used to make a new antibiotic. The mold is grown in a vat with a nutrient solution made up of sugars, water, and other ingredients. The laboratory makes two tests by introducing a known amount of the mold and observing how the mold grows over a 24 hour period. The results of these tests are shown in the table below.

Trial	Initial Population Size (p)	Population Size After 24 Hours	Growth Factor (r)	Reciprocal Factor (1/r)
A	100	200	2.0	
B	500	800	1.6	

1. Compute reciprocal growth factors. For each trial compute $1/r$ and enter the result in the final column of the table.

2. Linear Equation for $1/r$. The two trials in the table provide two points of the form (population size, reciprocal factor) = $(p, 1/r)$. Find the equation of the straight line through these two points. Express your equation in the form $1/r = mp + b$ where m is the slope and b is the intercept of the straight line.

3. Take the reciprocal of each side of the equation you found in step 2. Record the result below in the form of an equation for r.

$r = $ _____

Then continue to page 2.

4. On page 1 you should have found the equation $r = \dfrac{1}{\frac{1}{3200}p + \frac{15}{32}}$
or something similar. By multiplying the right side of the equation by 3200/3200, it is
simplified to $r = \dfrac{3200}{p + 1500}$. Verify that the r and p values in the table in step 1 satisfy this
equation.

5. The equation for r in the preceding step can be used to chart the future growth of the
population. For example, with $p_0 = 1000$ we find $r = 3200/2500 = 1.28$. That means the
population after 24 hours will be $p_1 = 1.28 \cdot 1000 = 1280$. Repeating the process, with $p = 1280$
we find $r = 3200/2780 \approx 1.151$, leading to $p_2 \approx 1473$.

Follow this approach to find the next several terms p_n, recording your results in the table below.

number of 24 hour periods	population size	r for next entry
0	1000	1.28
1	1280	1.151
2	1473	
3		
4		
5		
6		
7		
8		
9		

6. The calculation of population values in the table above can be expressed succinctly in the
equation $p_{n+1} = p_n / (0.0003125 p_n + 0.46875)$.
The *diffeqn* tab of the excel file Primus-Logistic-Models.xlsx
available at http://www.dankalman.net/logistic/primus/
can be used to obtain a table and a graph of the p
values. The figure at right illustrates how to enter the
equation and starting population. Use the excel file to check your table entries above, and to see
the shape of the graph. Describe below what this model predicts for the future growth of the
mold population.

Initial value = 1000

$p_{n+1} = p_n / (\boxed{0.0003125}\ p_n\ +\ \boxed{0.46875})$

Refined Logistic Population Model Worksheet 3

For a sample of *E. coli* bacteria cultured in a fermenter, population growth data were reported by Dr. Michael Blaber in a Molecular Biology class[1]. Population size was measured using absorbance of light, defined as $-\log_{10}$(fraction of light absorbed). For example, if a sample absorbs half of the incident light, the absorbance is $-\log_{10}(0.5) \approx 0.301$. The data are shown in the table and graph below.

Time (minutes)	Absorbance
0	0.090
18	0.102
78	0.124
142	0.253
205	0.487
255	1.020
322	1.980
378	3.950
446	5.880
504	6.760
564	7.200
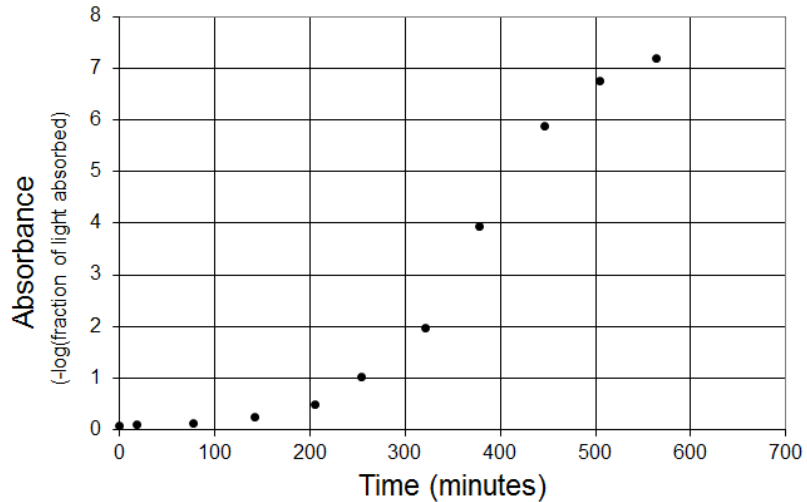
1. Visualize a logistic curve that approximately fits the data above. Estimate the location of the *y* intercept y_T and the inflection point, and record them below. [Your y_T must be greater than zero.]

y intercept (y_T) = _____ Inflection point (x_F, y_F) = _____ , _____

2. The equation for a logistic curve can be expressed in the form $y = \dfrac{A}{1+Br^x}$ with numerical constants in place of *A*, *B*, and *r*. It is known that *A* is equal to $2y_F$. Using your value of y_F from step 1, record the corresponding value of *A*.

$A =$ _____

3. Using the equation given in step 2, show that the *y* intercept of the curve is given by $y_T = \dfrac{A}{1+B}$. Then substitute the values of *A* from step 2 and y_T from step 1, and find the value of *B*. Show your work, and enter the value of *B* at the bottom. Then continue to page 2.

$B =$ _____

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[1] Dr. Michael Blaber, *Growth in Bacterial Populations* http://www.mikeblaber.org/oldwine/bch5425/lect12/lect12.htm

4. Using the equation $y = \dfrac{A}{1+Br^x}$, Show that $y = A/2$ when $Br^x = 1$. This corresponds to the inflection point, so $Br^{x_F} = 1$. Substituting the values you found earlier for B and x_F, find the value of r. Show your work and record the answer to four decimal places. [Hint: $r = (1/B)^{(1/x_F)}$].

$r =$ _____

6. The *logistic function* tab of the excel file Primus-Logistic-Models.xlsx available at
http://www.dankalman.net/logistic/primus/
can be used to obtain a table and a graph of the data as shown in the figure below.

Logistic Function

$$y = \dfrac{A}{1 + B \cdot R^x}$$

Fit a logistic function to data points by editing the yellow cells.

$y = \dfrac{\text{Put your value of } A \text{ here}}{1 + \boxed{\text{your } B} \cdot \boxed{\text{your } r}}$ x

Lowest $x =$ 0   Highest $x =$ 700

Data Points		y from logist. equation	error = |eqn y - data y|
x	y		
0.00	0.09	0.1	0.01
18.00	0.10	0.119593103	0.017593103
78.00	0.12	0.216429768	0.092429768
142.00	0.25	0.403889366	0.150889366
205.00	0.49	0.734542269	0.247542269
255.00	1.02	1.158547398	0.138547398
322.00	1.98	2.044172356	0.064172356
378.00	3.95	3.108619057	0.841380943
446.00	5.88	4.718622452	1.161377548
504.00	6.76	6.154427891	0.605572109
564.00	7.20	7.452182005	0.252182005

Average absolute error = 0.325607897

Enter the x and y values in the yellow cells of the table, and your values of A, B, and r in the yellow cells above the right half of the graph. How well does the curve you see approximate the data points? By trial and error, adjust your A, B, and r values so that the curve comes as close as possible to all the points. Record your best values of A, B, and r and the average absolute error for the corresponding curve below.
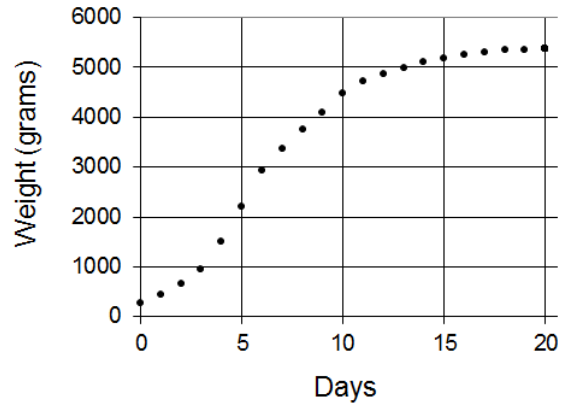
A _____ B _____ r _____   Average absolute error _____

Refined Logistic Population Model Worksheet 4

The table and graph below show data related to the growth of one fruit on a *cucurbito pepo* plant, which may have been a squash or pumpkin.[1] As the fruit was growing, its weight was measured daily. On this work sheet you will develop a refined logistic growth model based on the data.

Time (days)	Weight (grams)	Time (days)	Weight (grams)
0	267	11	4720
1	443	12	4864
2	658	13	4980
3	961	14	5114
4	1498	15	5176
5	2200	16	5242
6	2920	17	5298
7	3366	18	5352
8	3758	19	5360
9	4092	20	5366
10	4488		

1. Reciprocal Growth Factors. In developing a refined logistic growth model for the data above, we will need to work with the reciprocal growth factors for the data. These are defined as follows. For any weight (w) the reciprocal growth factor is defined by $1/r = weight / next\ day's\ weight$. Find the reciprocal growth factor for each of the weights in the table at left below, using the data at the top of the page. The first row of the table has been completed by computing $1 / r = 267/443$.

| w	$1 / r$
267	0.6027
2920	
4980	
5360	
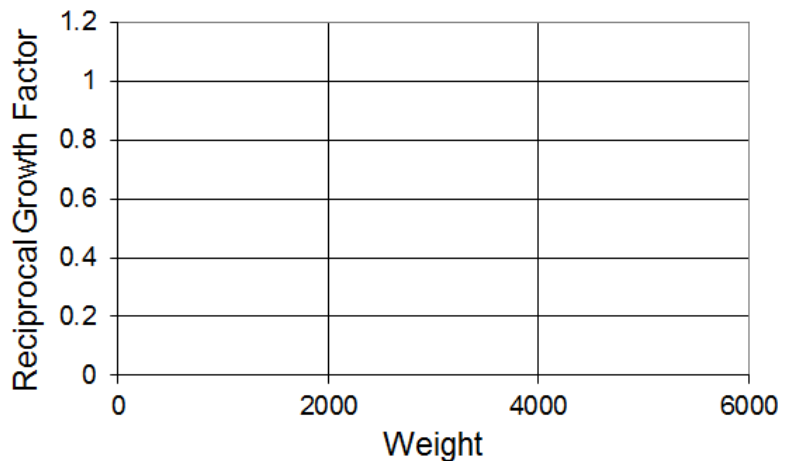
2. Plot your results on the graph next to the table, and draw a straight line that comes as near to all the points as possible. Then find the equation of your line. If your line goes through two of your points, you can find the equation using those points. Otherwise estimate the slope and intercept.

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[1] Reported in "The Chemical Basis of Growth and Senescence", by T. Brailsford Robertson, J. P. Lippencott, Philadelphia and London,1923, pp72-74, with an attribution to A. P. Anderson, 1895.

For a more accurate model, we should use all of the data from the top of the preceding page. The reciprocal growth factors are shown in the tables below for all of the available data, and the values of $1/r$ are plotted versus w in the adjoinging graph

w	$1/r$
267	0.602709
443	0.673252
658	0.684703
961	0.641522
1498	0.680909
2200	0.753425
2920	0.867499
3366	0.895689
3758	0.918377
4092	0.911765

w	$1/r$
4488	0.950847
4720	0.970395
4864	0.976707
4980	0.973797
5114	0.988022
5176	0.987409
5242	0.98943
5298	0.98991
5352	0.998507
5360	0.998882
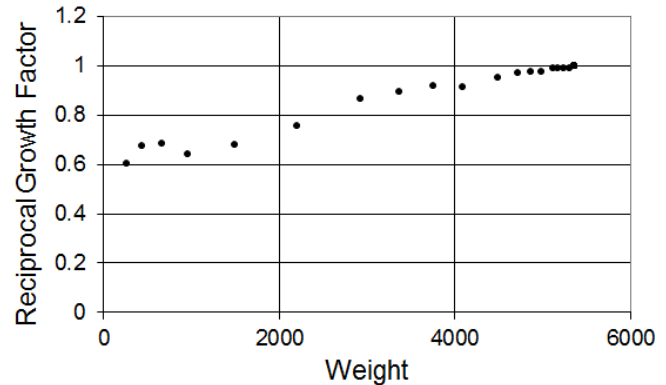
3. Find the equation for a line that comes as close as possible to all of the points in the graph above. The *Find m and b* tab of the excel file Primus-Logistic-Models.xlsx available at http://www.dankalman.net/logistic/primus/ can be used for this purpose. It will allow you to experiment with different values of the slope and intercept to obtain a line that comes close to the points. Record your best choice for slope and intercept below.

slope = _____   intercept = _____

4. Write the equation for your line below. However, because the horizontal axis of your graph represents weight and the vertical axis represents reciprocal growth factor, write your equation using w in place of x and $1/r$ in place of y.

_____

5. Take the reciprocal of both sides of your equation from part 4 to express an equation in the form

$$r = \frac{1}{mw+b} .$$

_____

6. The refined logistic growth difference equation has the form $w_{n+1} = \dfrac{w_n}{mw_n+b}$. Using your values of m and b from part 5, and $w_0 = 267$, use the difference equation to compute the next five w values, and compare the results with the given data. Show your results below. How well does your difference equation approximate the original data?

7. The *diff eqn + data points* tab of the excel file Primus-Logistic-Models.xlsx mentioned earlier can be used to improve the agreement between your difference equation and the data. With the weight data entered in the yellow cells under *data values*, and your coefficients and a value for w_0 entered in the appropriate yellow cells, you will see both the original data and the results of the difference equation both graphically and numerically. Using trial and error, you can alter your coefficients and w_0 to obtain better agreement between the difference equation and the original data. The figure below shows a screenshot using one set of coefficients. But you should enter the values you found on the preceding page.

Directions: hover mouse HERE

$$p_{n+1} = \frac{p_n}{mp_n + b}$$

Initial value = 250

Refined Logistic Growth

Study various refined logistic growth models by editing the yellow cells.

Difference Equation: $p_{n+1} = p_n / ($ 0.0001 $p_n +$ 0.65 $)$ Lowest $n =$ 0   Highest $n =$ 23

Avg. error: 1222.9846

position number	data values	terms based on diff eqn
0	267	250
1	443	370.3704
2	658	539.0836
3	961	765.8434
4	1498	1054.032
5	2200	1395.324
6	2920	1767.279
7	3366	2137.679
8	3758	2474.831
9	4092	2757.523
10	4488	2978.684
11	4720	3142.508
12	4864	3259.015
13	4980	3339.492
14	5114	3393.968
15	5176	3430.34

Record your best values of w_0, m, and b and the average absolute error for the corresponding model below.

w_0 _____ m _____ b _____   Average absolute error _____