Ellipses . . .

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Topics

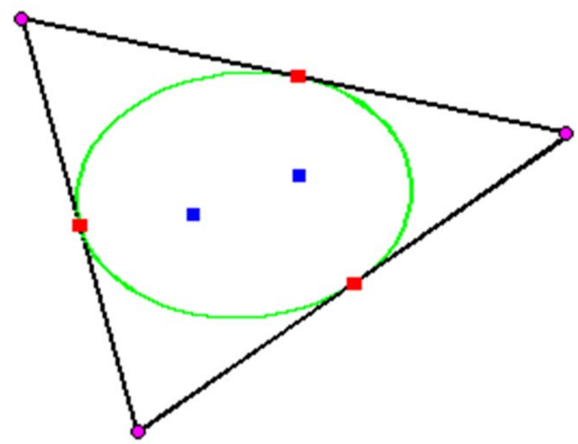
- Marden's Theorem and Mine Safety
- The Ladder Problem, Astroidal Mesh
- Ellipsoidal Runners in the Rain
- Deflection on an Ellipse (Time Permitting)

Marden's Theorem
and
Mine Safety

Marden's Theorem

- Mentioned in Pam Gorkin's Falconer Lecture
- Cubic polynomial p over $\mathbb{C}$
- Given the roots of p, locates roots of p'
- Geometrically: foci of the Steiner In-Ellipse

- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci
- Those are the roots of $p'(z)$



Corollary

- Fact: The mean of the roots of polynomial p equals the mean of the roots of p'.
- In Marden's Theorem mean of roots = centroid of the triangle = mean of the roots of p'.
- So roots of p' are symmetric about the centroid.
- Center of ellipse is at the centroid

Dynamic Geometry

- Click and drag point *A*.
- See both *A* and its reflection *B* across the centroid
- See three ellipses with foci at *A* and *B*. Each ellipse passes through the midpoint of one side
- Goal: inscribed ellipses
- Demo 1
- Demo 2: show the value of $p'(A)$

Mine Safety

- Email out of the blue from Monte Hieb, Chief Engineer, WV Office of Miners' Health, Safety, and Training
- "For a triangle whose 3 vertices are known, how does one determine the angle of inclination of the major axis of the triangle's Steiner Ellipse?"
- His motivation: "characterization of stress-strain ellipses for safety enhancement in underground mining applications" .
- He built an excel spreadsheet for field inspectors. They entered data and the spreadsheet computed the axis of the stress-strain ellipse.

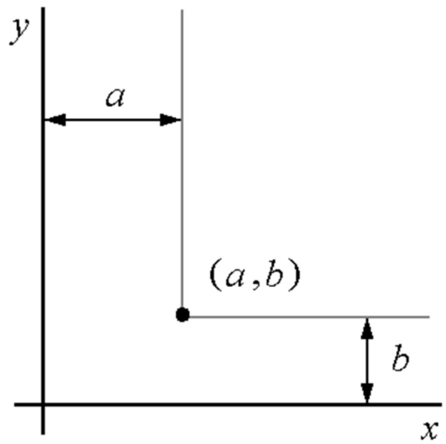
Solution with Marden's Theorem

- Given vertices: (x_1, y_1), (x_2, y_2), (x_3, y_3)
- Complex numbers: $z_k = x_k + iy_k$
- Let $p(z) = (z - z_1)(z - z_2)(z - z_3)$
- Compute $p'(z)$ in form $3z^2 + bz + c$
- Find roots with quadratic formula
- Express as points in the plane
- They are foci, so determine the major axis.
- Other solutions exist w/o Marden's Theorem

The Ladder Problem and Astroidal Mesh

The Ladder Problem:

How long a ladder can you carry around a corner?



To Review…

- Seen in Gregory Quenell’s talk in this session
- Slide the ladder around the corner, keeping its ends touching outer walls
- The moving ladder sweeps out a region Ω
- The boundary curve for Ω is a well known curve from geometry: an *astroid*, aka a *hypocycloid of four cusps*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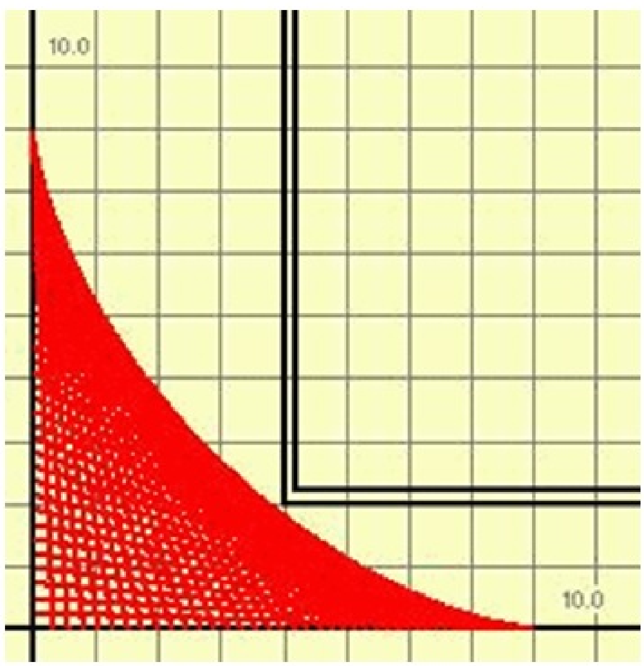
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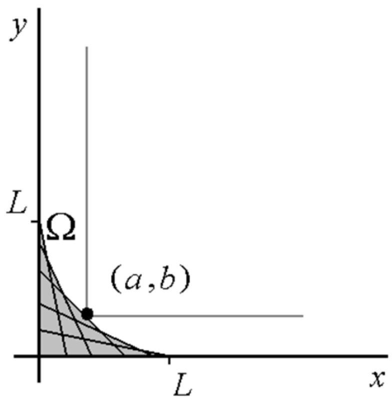
Solution to Ladder Problem

- Ladder will fit if (a,b) is outside the region Ω
- Ladder will not fit if (a,b) is inside the region
- Longest L occurs when (a,b) is on the curve:

$$a^{2/3} + b^{2/3} = L^{2/3}$$

$$L = (a^{2/3} + b^{2/3})^{3/2}$$

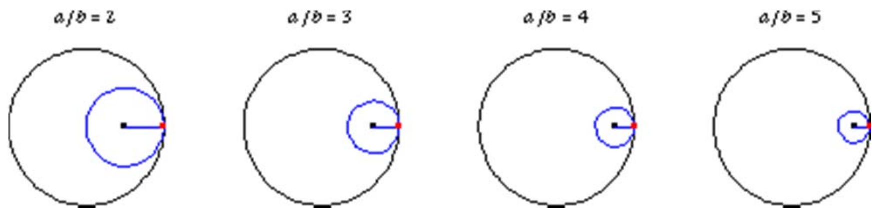


Where do ellipses come in?

- Astroid plays a central role
- It arises as envelope for a family of lines
- Among many other interesting properties…
- It also arises as an envelope of a family of ellipses

A famous curve

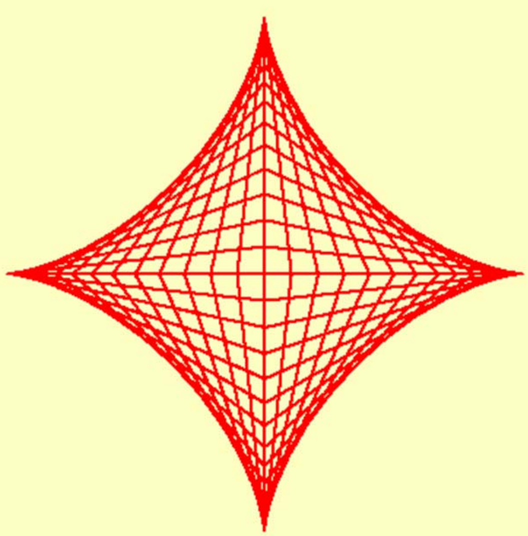
Hypocycloid: point on a circle rolling within a larger circle



Astroid: larger radius four times larger than smaller radius

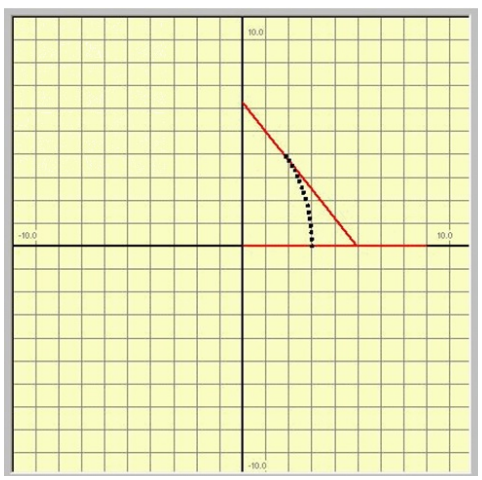
Animated graphic from Mathworld.com

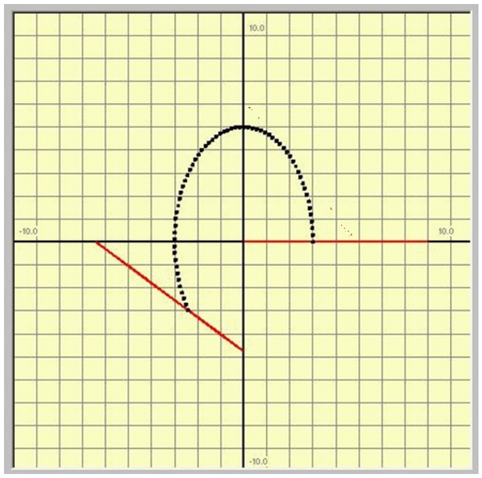
Astroidal Mesh



Alternate View

- Ellipse Model: slide a rigid line segment of length L with its ends on the axes, like our ladder
- Let a fixed point on the segment trace a curve
- The traced curve is an ellipse
- The fixed point divides the segment in two parts, with lengths a and b, so $L = a + b$
- The traced ellipse has semi-axes a and b
- Animation on next slide

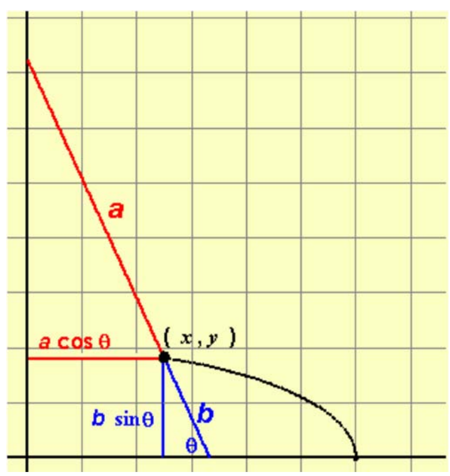




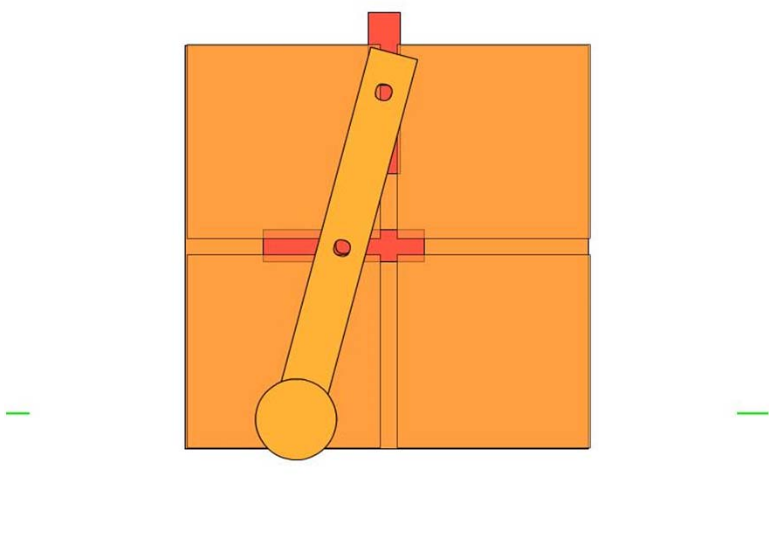
Why is the curve an ellipse?

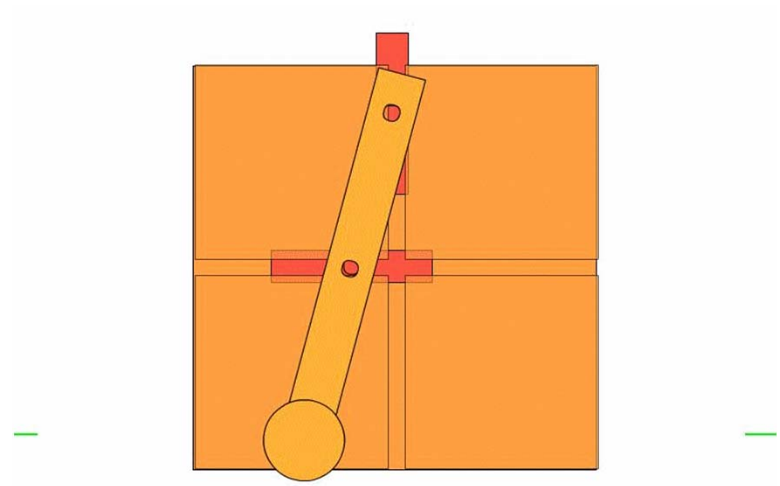
- Let θ = angle made by ladder and x axis
- $x = a \cos \theta$
- $y = b \sin \theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Trammel of Archimedes





Family of Ellipses

- Paint an ellipse with *every* point of the ladder
- Family of ellipses with sum of major and minor axes equal to length *L* of ladder
- These ellipses sweep out the same region as the moving line
- Same envelope

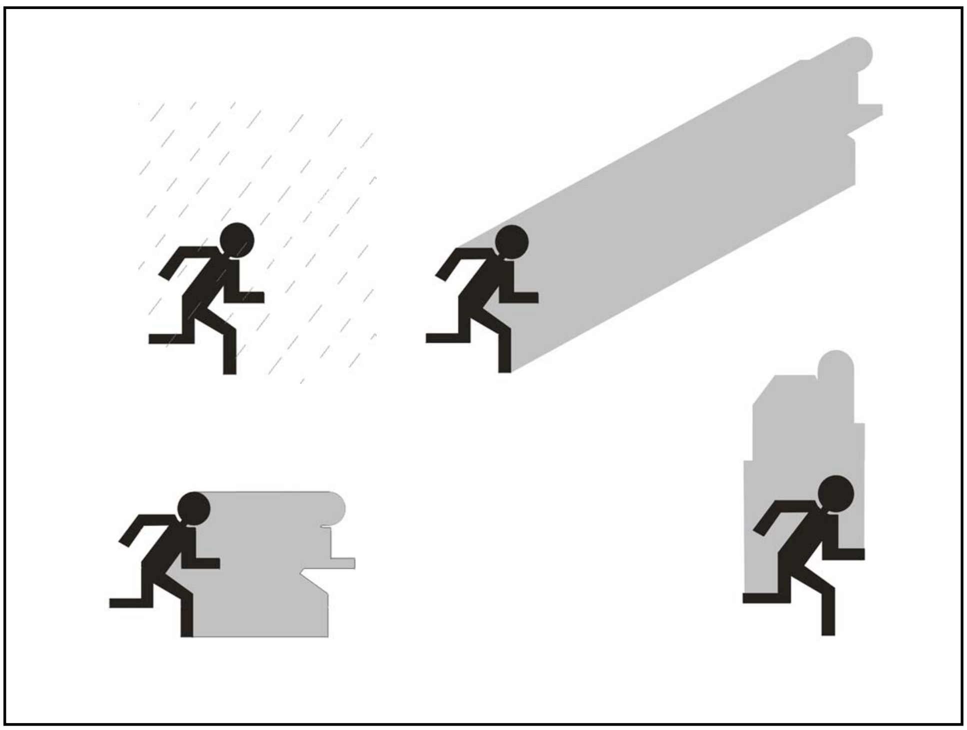
[Mesh Demo 1]

[Cool Java Applet]

Ellipsoidal Runners
Out in the Rain

2D Version and Rain Regions

- Problem: what pace minimizes the amount of rain that hits a runner
- 2D Geometric Approach
- Assume constant rate and direction of rainfall
- Find possible initial positions from which drops can hit the runner
- This defines the rain region
- Minimize incident rain by minimizing the area of the rain region.



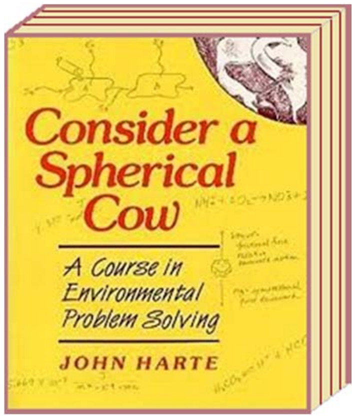
Analysis

- Assume runner is a rectangle
- Rain region is a parallelogram
- Area easily computed
- Runner should go as fast as possible.

3D Case

- Analogous definition of rain region
- Obvious shape assumption: runner is a cereal box
- Rain direction can include both in-track and cross-track directions
- Optimal pace is *fast-as-possible* with a head wind or slight tail wind
- With a stronger tail wind it is best to match the speed of that wind. (Front and back of the cereal box stay dry.)

Ellipsoidal Runners



- Rectangular prism is a poor model for a runner
- One obvious alternative: a sphere
- This is a well known model for cows, who may also wish to stay dry
- Ellipsoids may be a more accurate model, and are no harder to analyze

Analysis

- For ellipsoidal runner
- Rain region is a cylinder swept out by translating the ellipsoid
- Amount of rain is proportional to the volume of the rain region
- Key computation: cross-sectional area of rain region
- Question: if an ellipsoid with semi-axes a, b, c is projected along vector $\mathbf{v}$ onto the orthogonal plane P, what is the area of the projection?

Ellipsoid Projection Problem

- This question can be formulated and answered in n dimensions as easily as in 3
- The solution has an appealing simplicity
- It permits us to solve the rain problem for ellipsoidal cows in n dimensions.

Ellipsoid Projection Theorem

- Let E be the n dimensional ellipsoid with equation $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1$
- Let U_n be the n-volume of the unit sphere in $\mathbb{R}^n$
- Let **v** be the position vector of a point on E.
- Project E on an n – 1 dimensional hyperplane orthogonal to **v**
- Projection has (n – 1) – volume

$$\frac{U_{n-1}}{\|\mathbf{v}\|} a_1 a_2 \cdots a_n$$

Deflection on an Ellipse and Geodetic Latitude

Deflection Demo

Homework Problems

- What is the maximum deflection?
- Where on the ellipse is the maximum attained?
- What about in *n* dimensions?
- What does this have to do with Geodetic Latitude?