Dunham's Morley Challenge

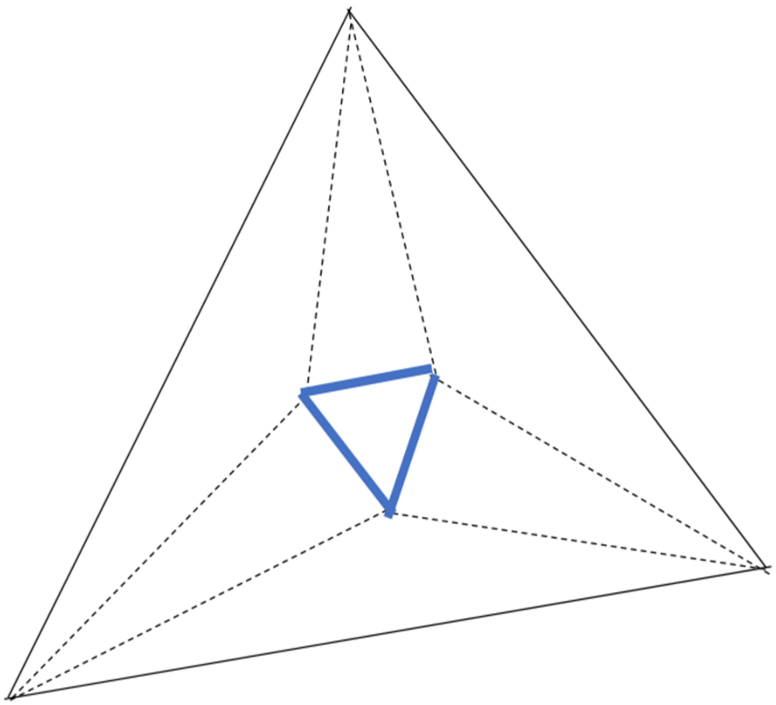
How Desmos almost let me rediscover Conway's amazing proof of Morley's Theorem

Dan Kalman • American University

www.dankalman.net

Morley's Theorem

- For an arbitrary triangle
- There are two trisectors of each angle
- For each edge, intersect the nearest trisectors from each adjacent vertex
- These are the vertices of an equilateral triangle.
- Call the resulting figure the ***Morley diagram*** for the given triangle

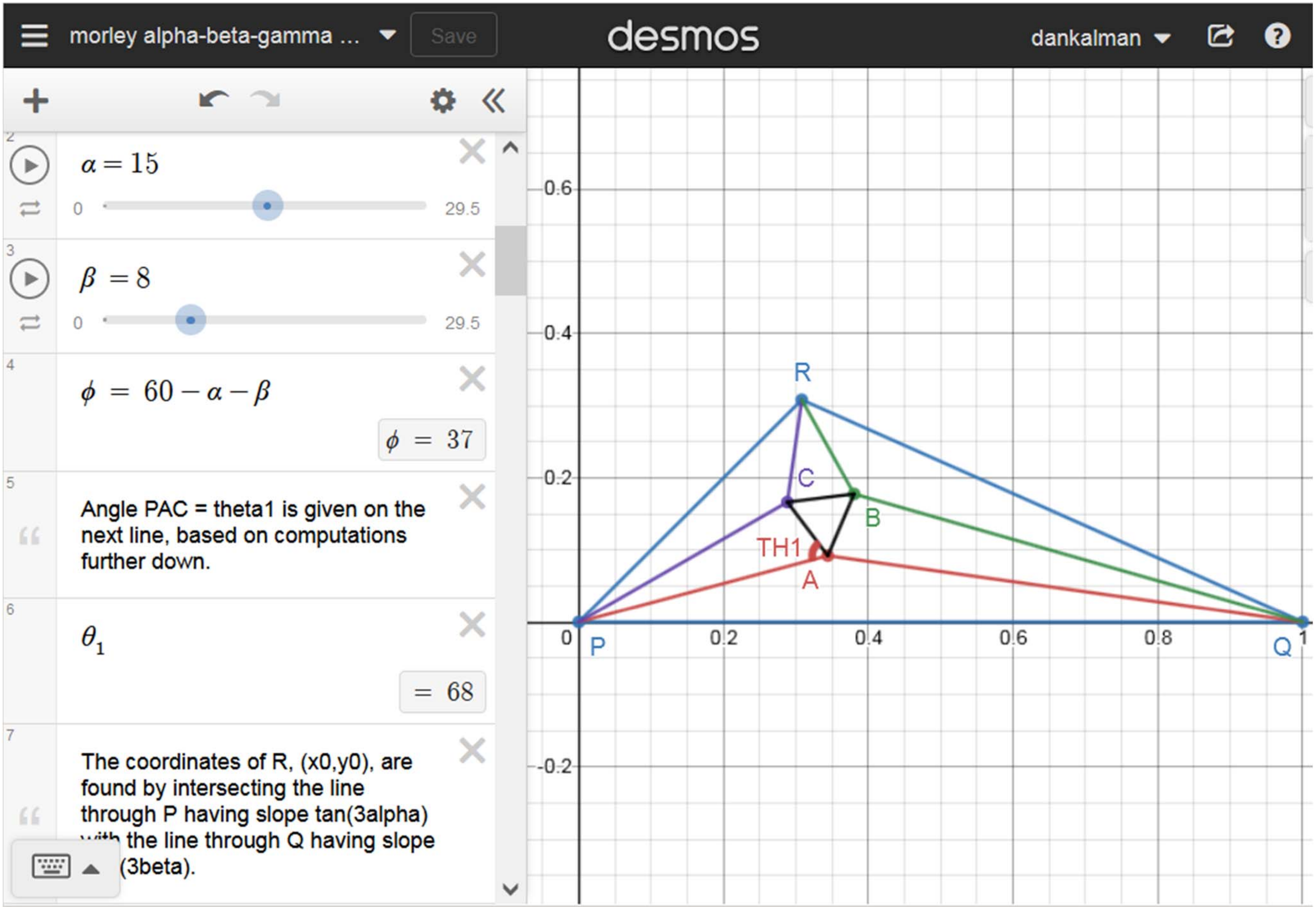


Dunham’s Criteria …

- … For an optimal proof
- Angles only
- Use angle relationships to deduce that the inner triangle is equi-angular
- Not reverse engineered

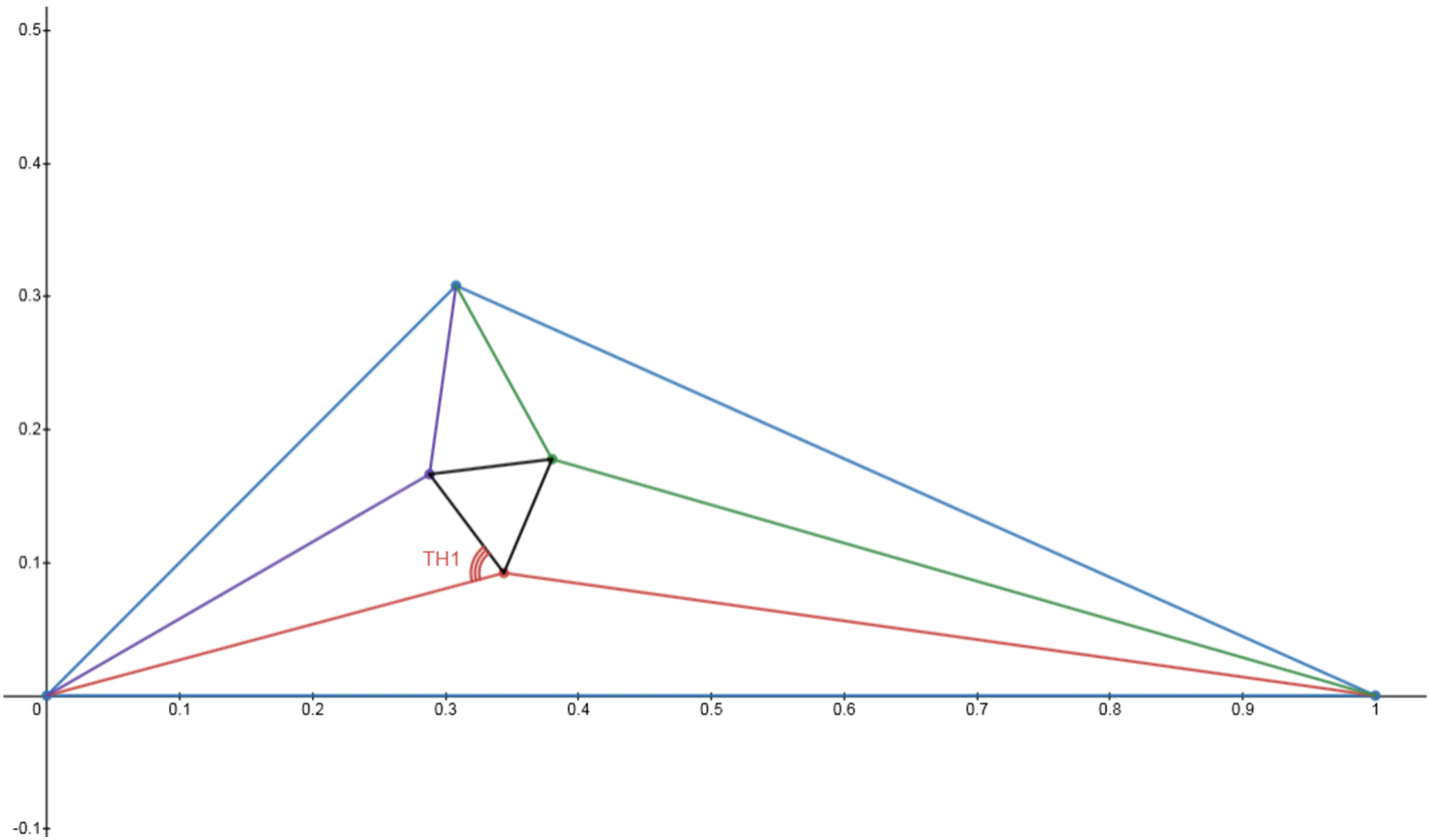
Desmos

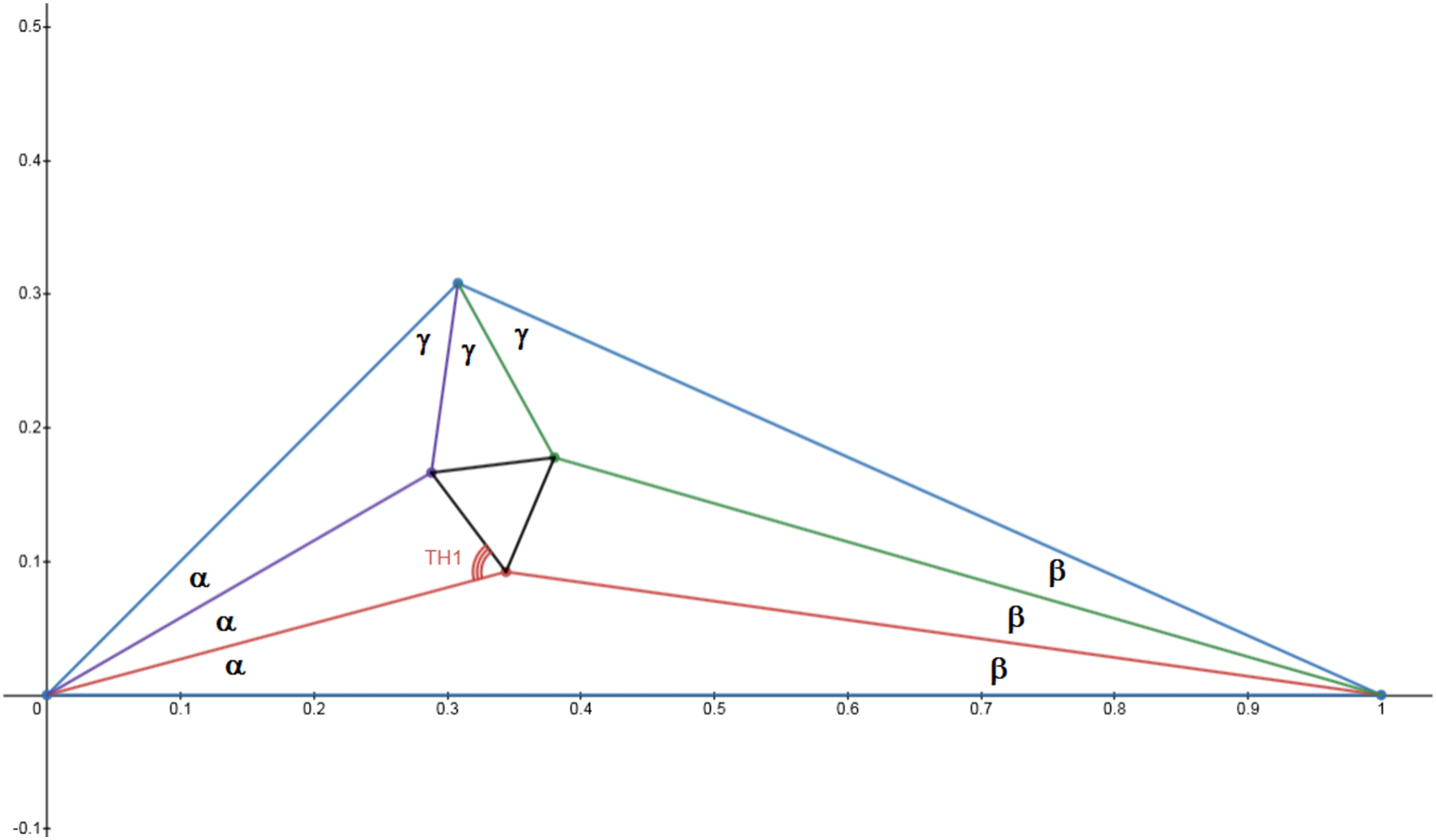
- To investigate angles, use Desmos to actually compute angles in a Morley diagram
- Assume angles of triangle are 3α, 3β, 3γ
- Assume 3α vertex at (0,0), 3β vertex at (0,1), and 3γ vertex in upper half plane
- Can compute vertices of inner triangle and hence determine the angles.
- Pyth thm for edge lengths, law of sines or cosines for angles


morley alpha-beta-gamma ...
Save
desmos
dankalman
α = 15
0
29.5
β = 8
0
29.5
ϕ = 60 − α − β
ϕ = 37
Angle PAC = theta1 is given on the next line, based on computations further down.
θ1
= 68
The coordinates of R, (x0,y0), are found by intersecting the line through P having slope tan(3alpha)
the line through Q having slope
(3beta).
R
C
B
TH1
A
P
Q
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
-0.2

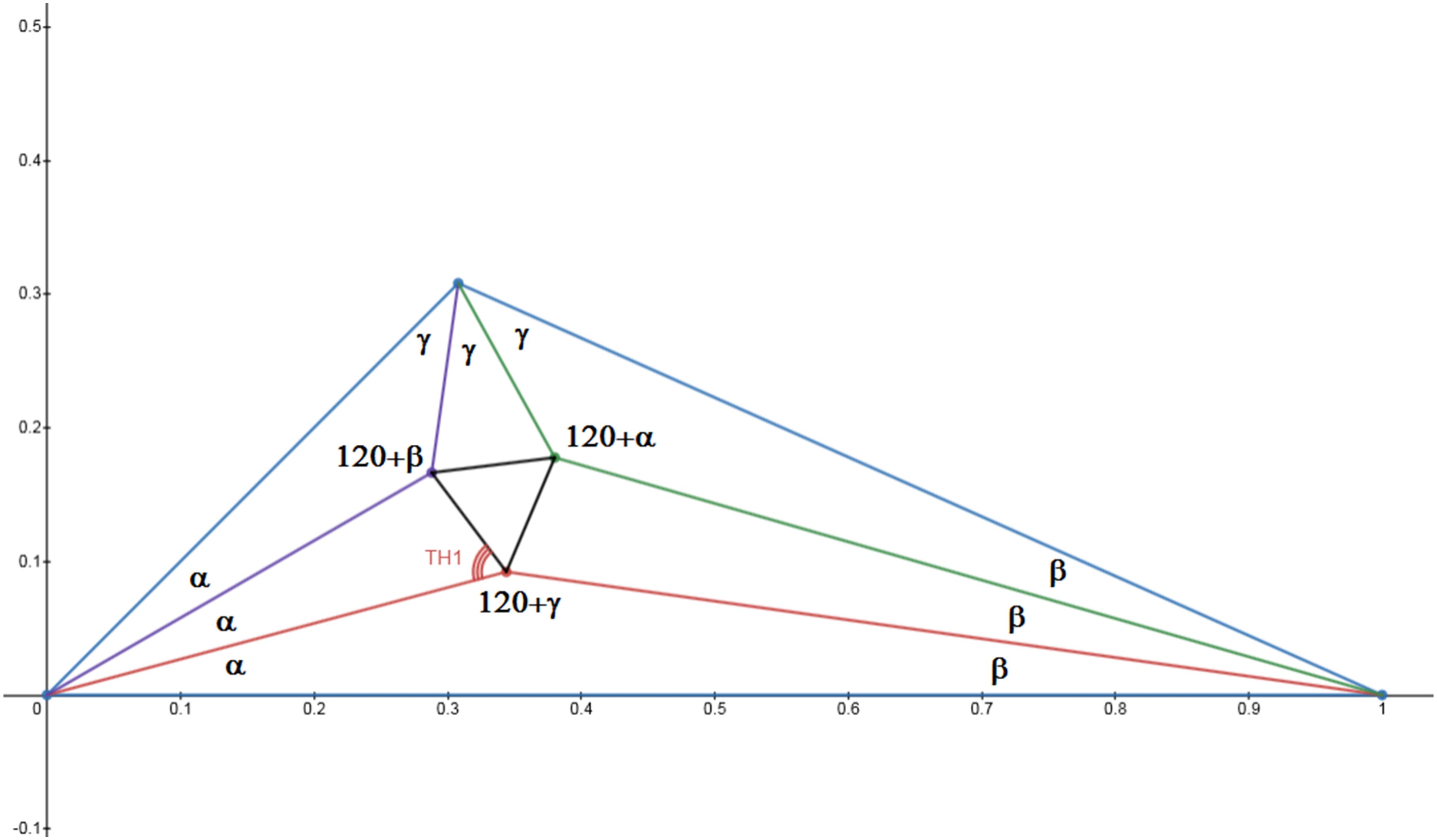
Load Desmos Page

https://www.desmos.com/calculator/grytmq8w0a

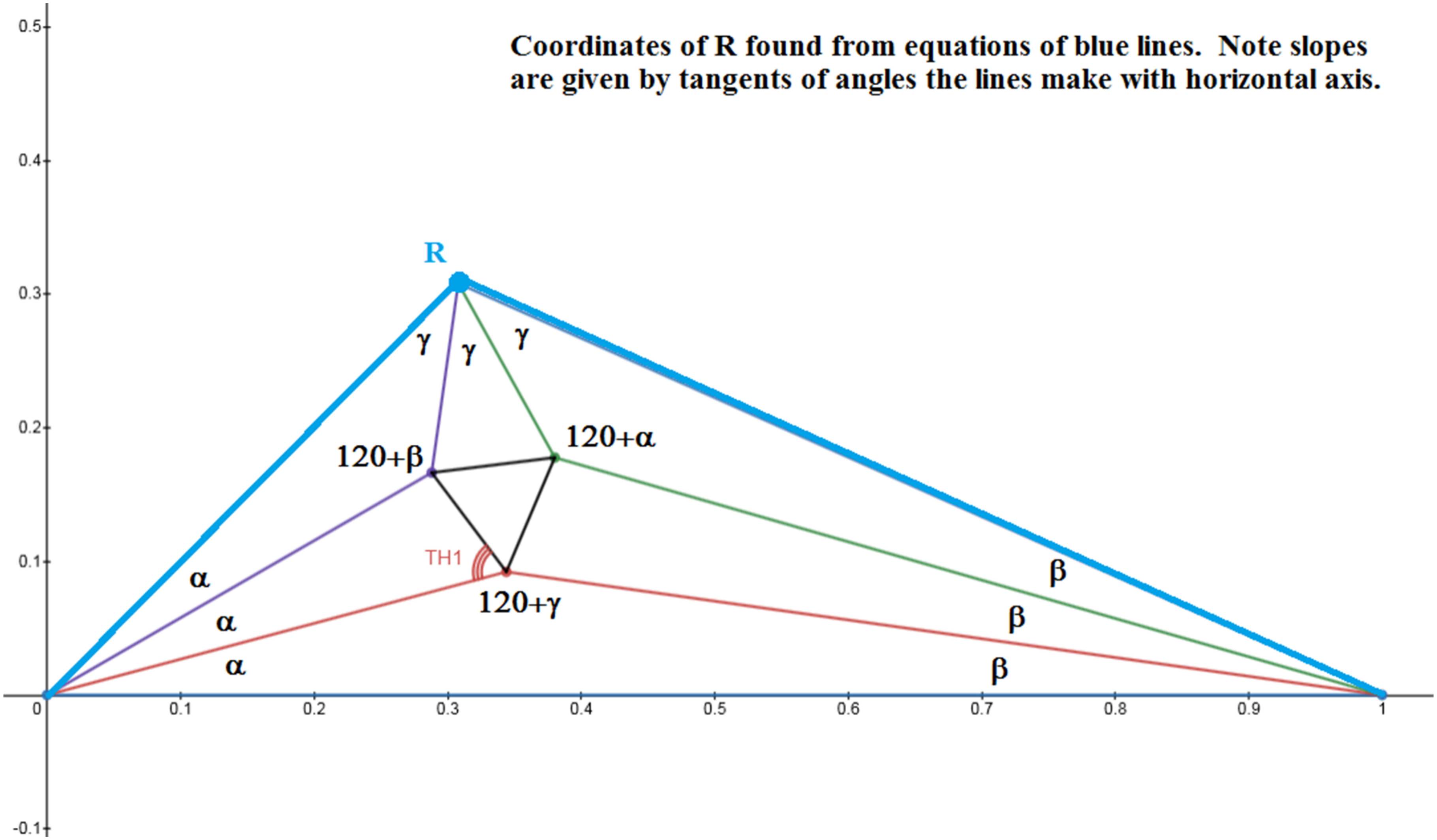

0.5
0.4
0.3
0.2
0.1
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
TH1



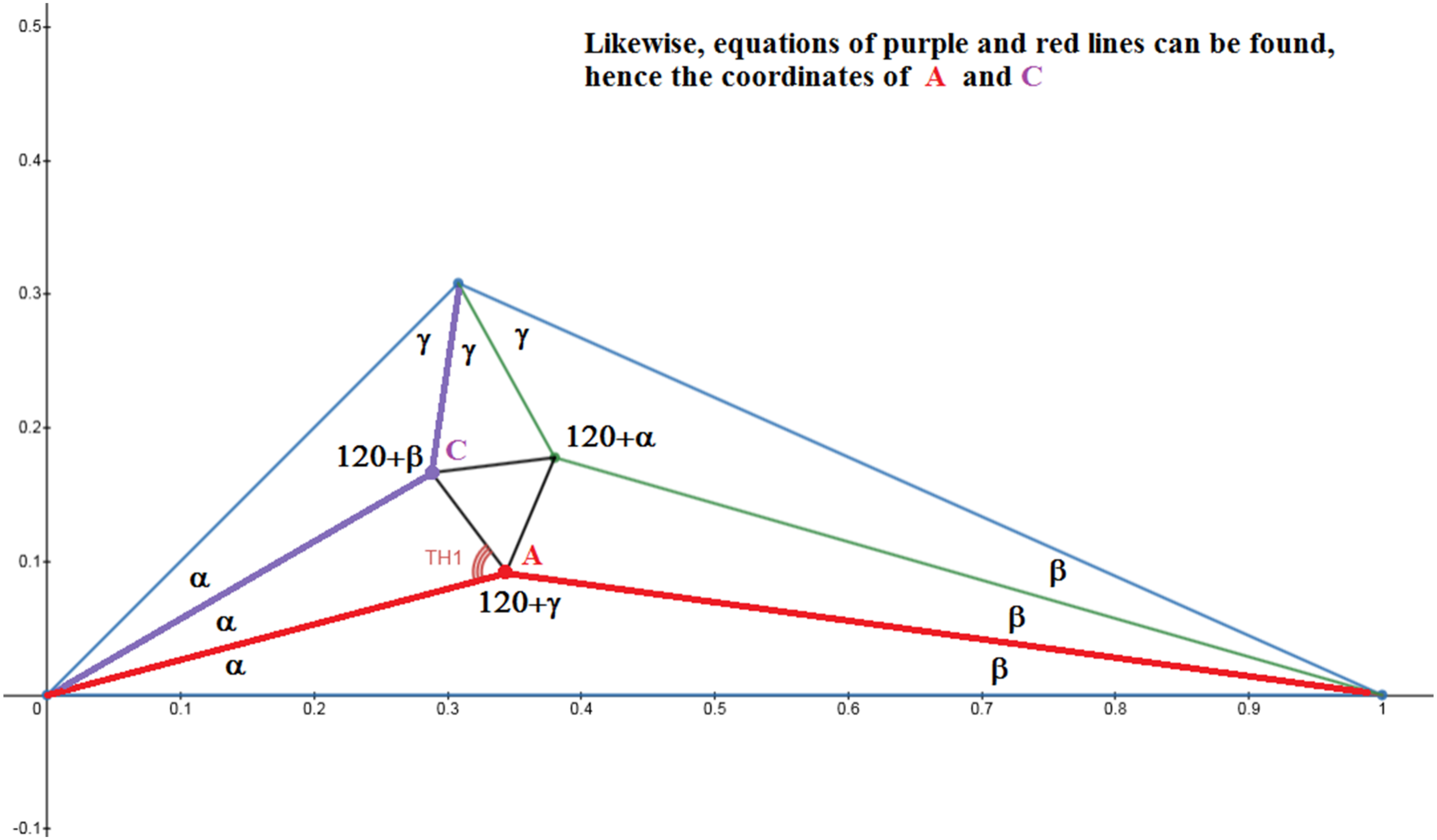
0.5
0.4
0.3
0.2
0.1
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
γ
γ
γ
α
α
α
β
β
β
TH1



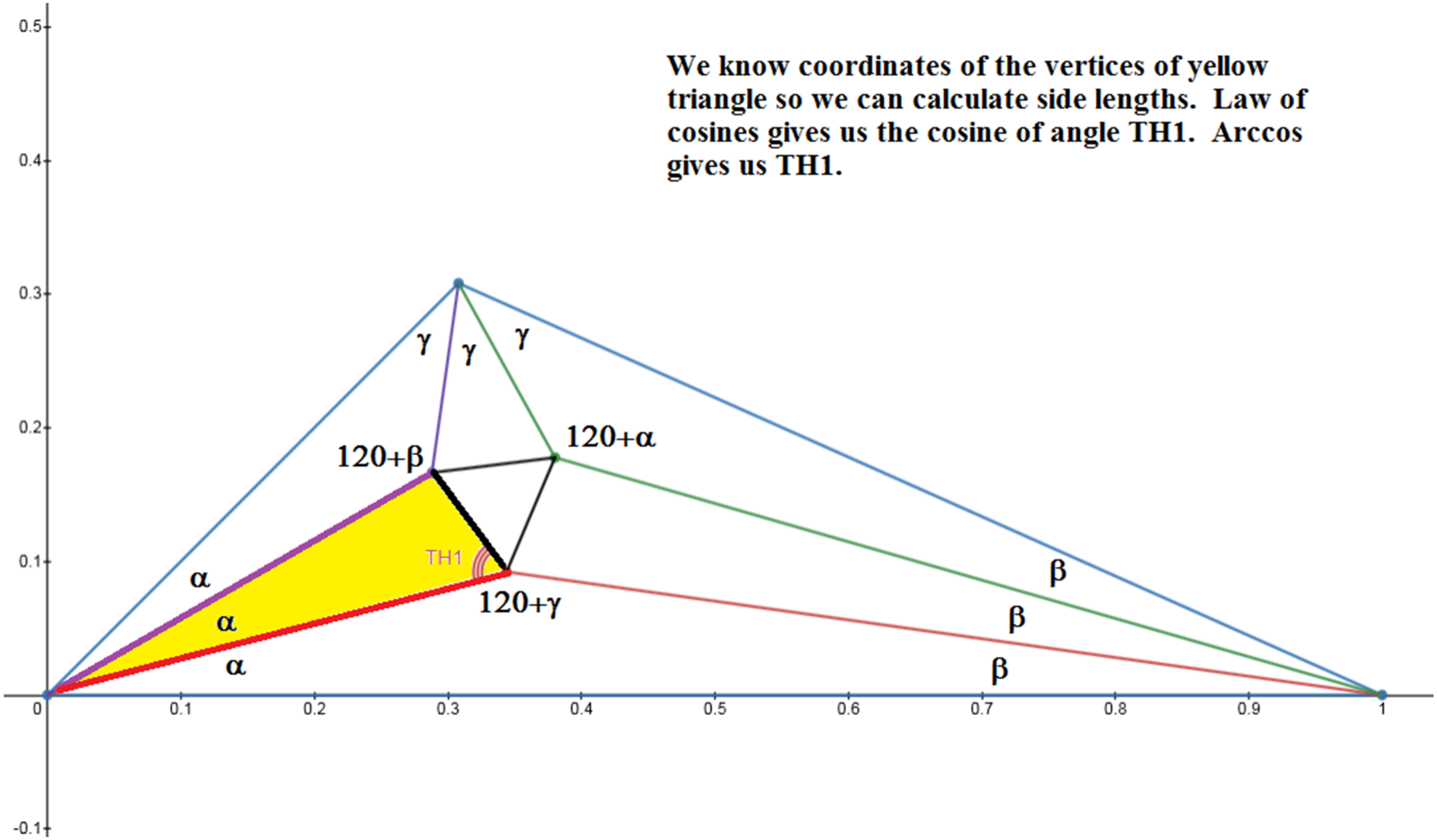
0.5
0.4
0.3
0.2
0.1
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
γ
γ
γ
120+β
120+α
TH1
120+γ
α
α
α
β
β
β

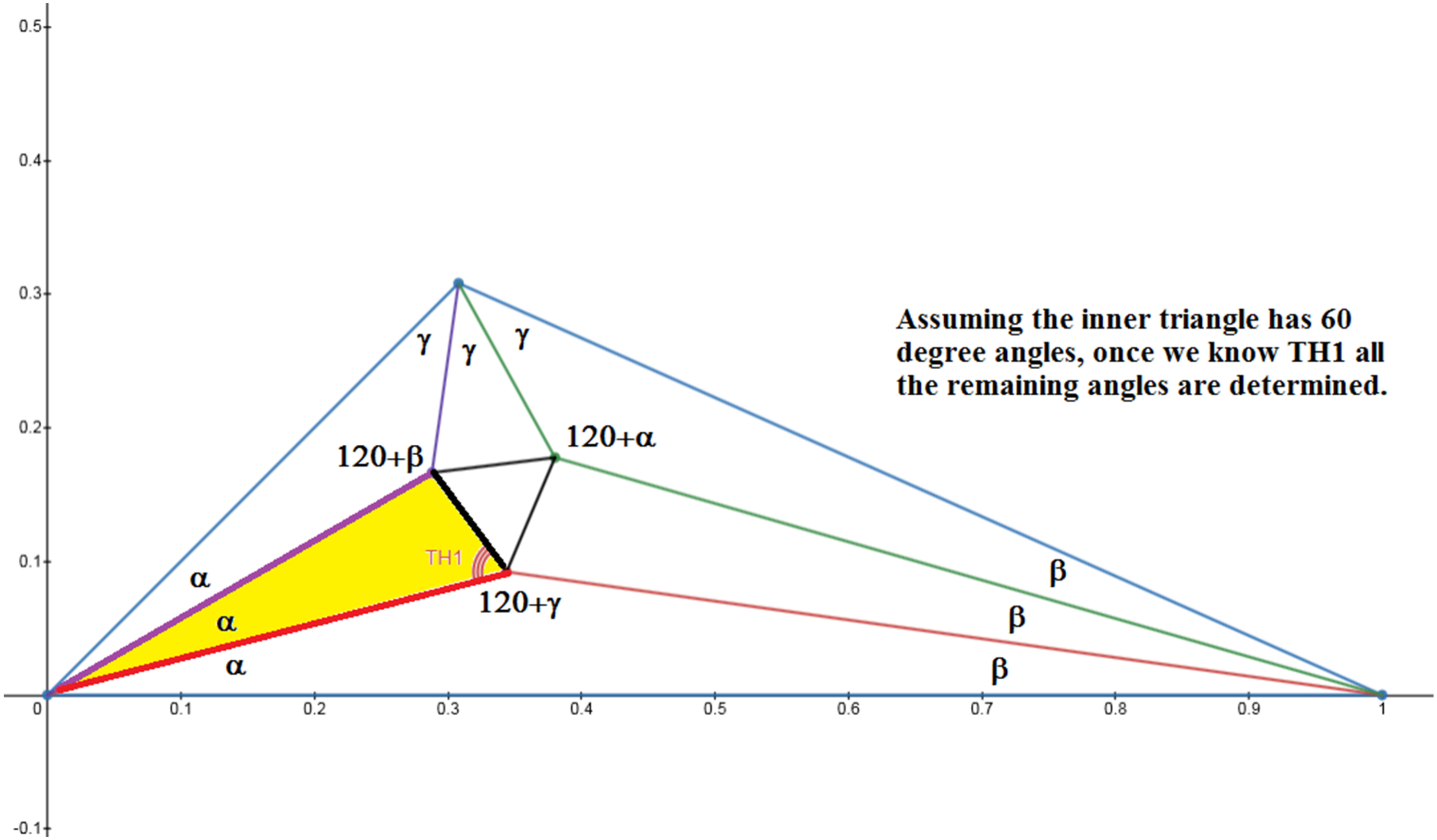

Coordinates of R found from equations of blue lines. Note slopes
are given by tangents of angles the lines make with horizontal axis.
R
γ
γ
γ
120+β
120+α
TH1
120+γ
α
α
α
β
β
β
0.5
0.4
0.3
0.2
0.1
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Likewise, equations of purple and red lines can be found, hence the coordinates of A and C



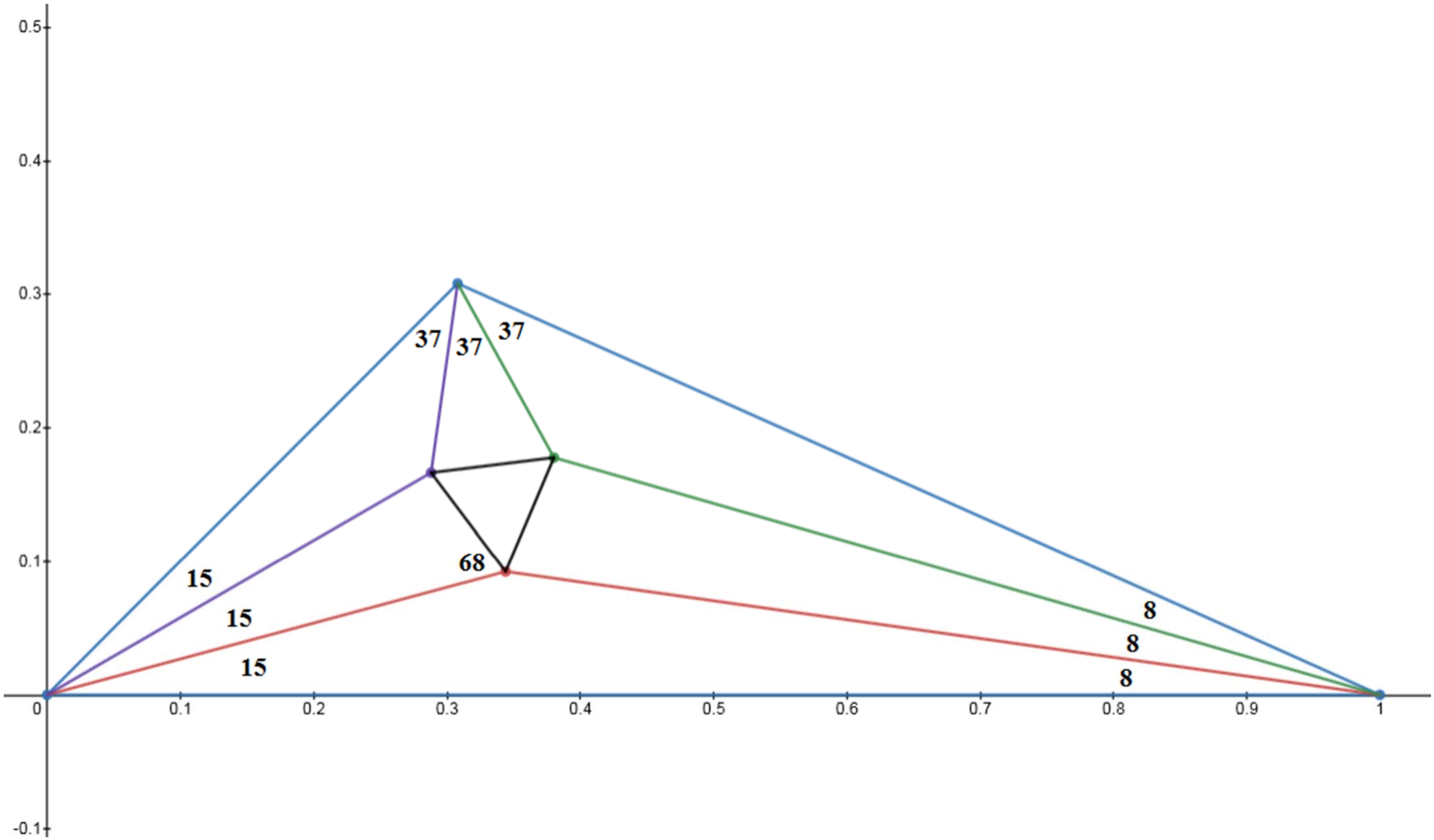
We know coordinates of the vertices of yellow triangle so we can calculate side lengths. Law of cosines gives us the cosine of angle TH1. Arccos gives us TH1.



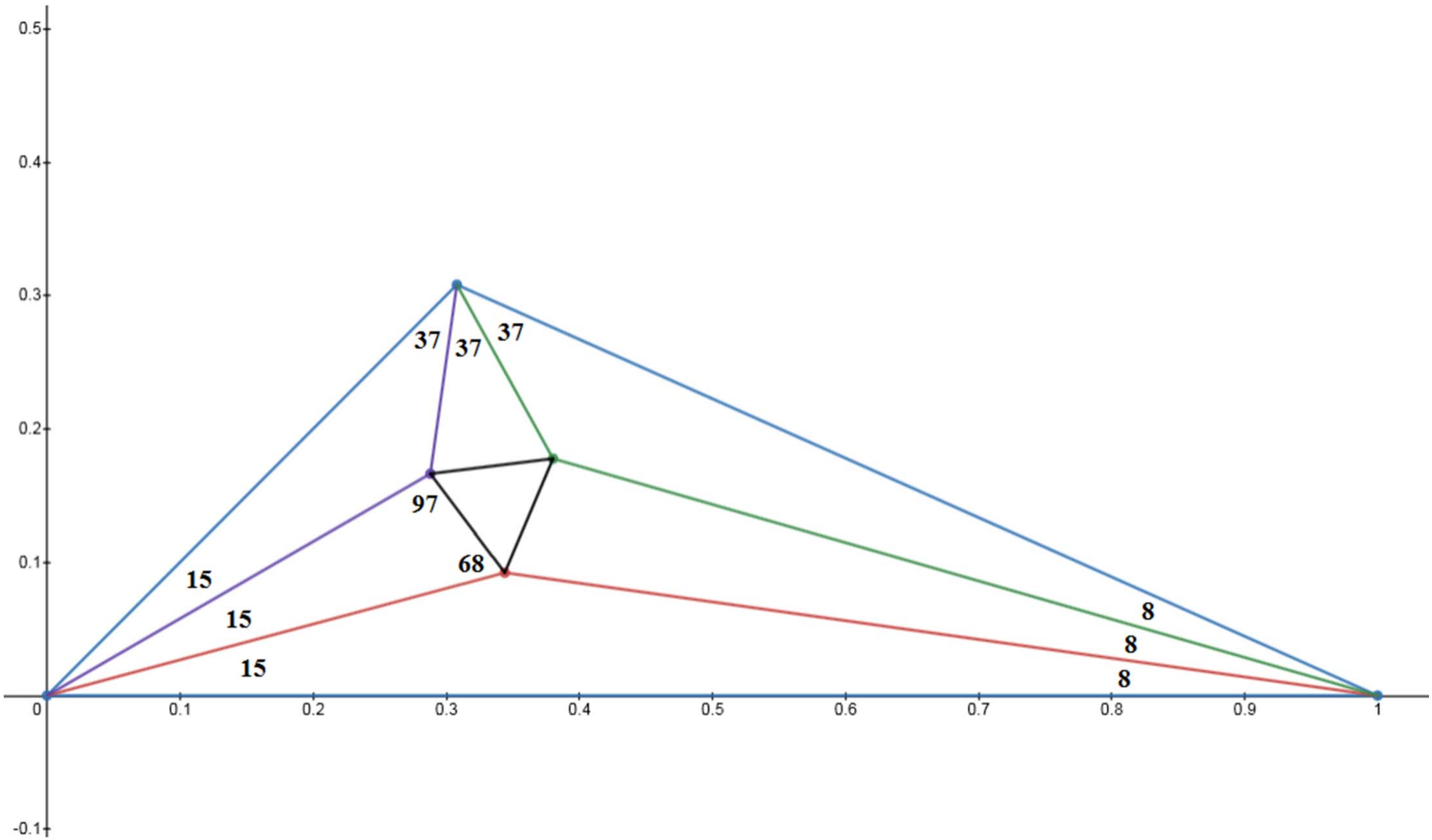


Assuming the inner triangle has 60 degree angles, once we know TH1 all the remaining angles are determined.

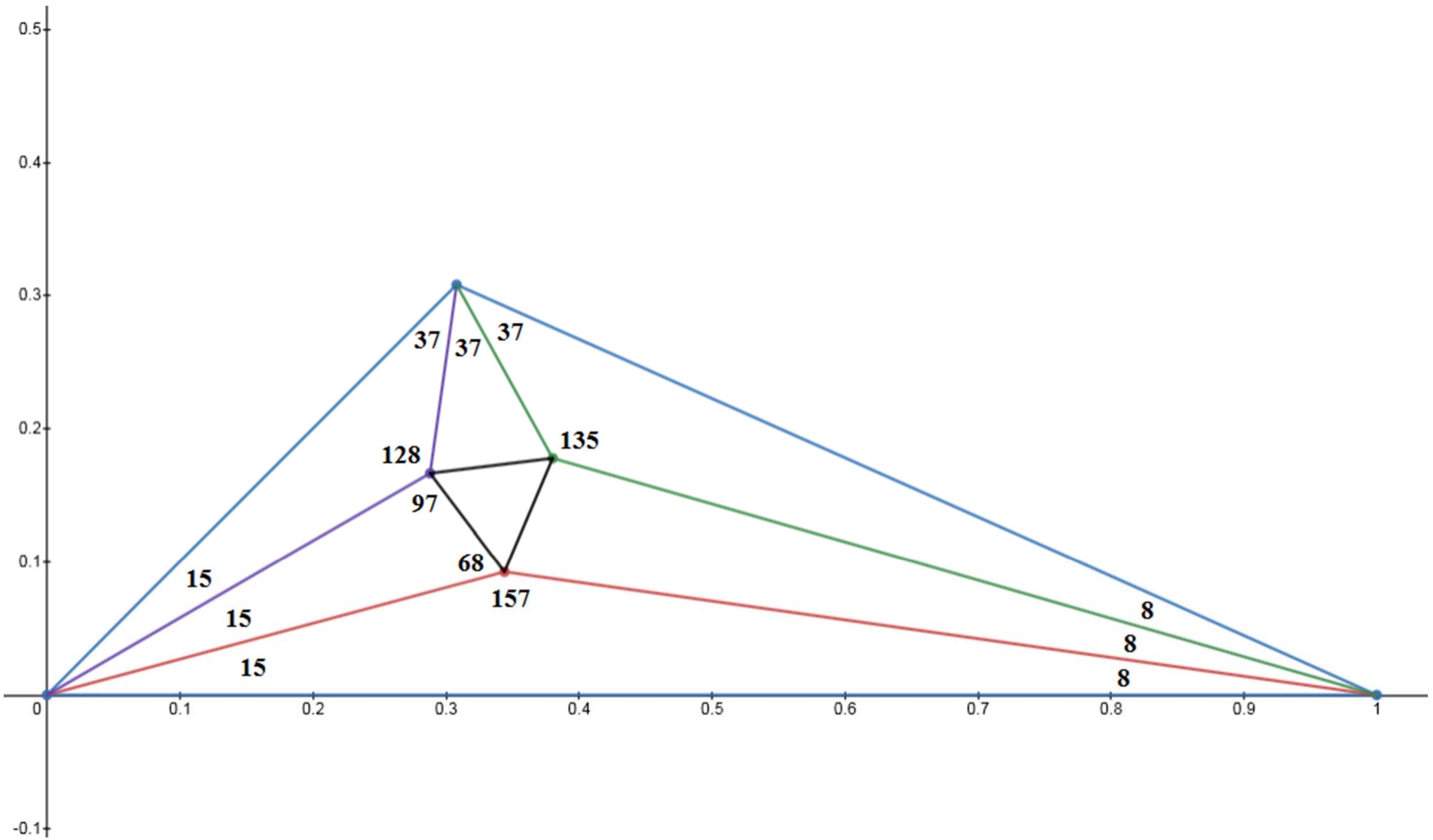
Earlier Example



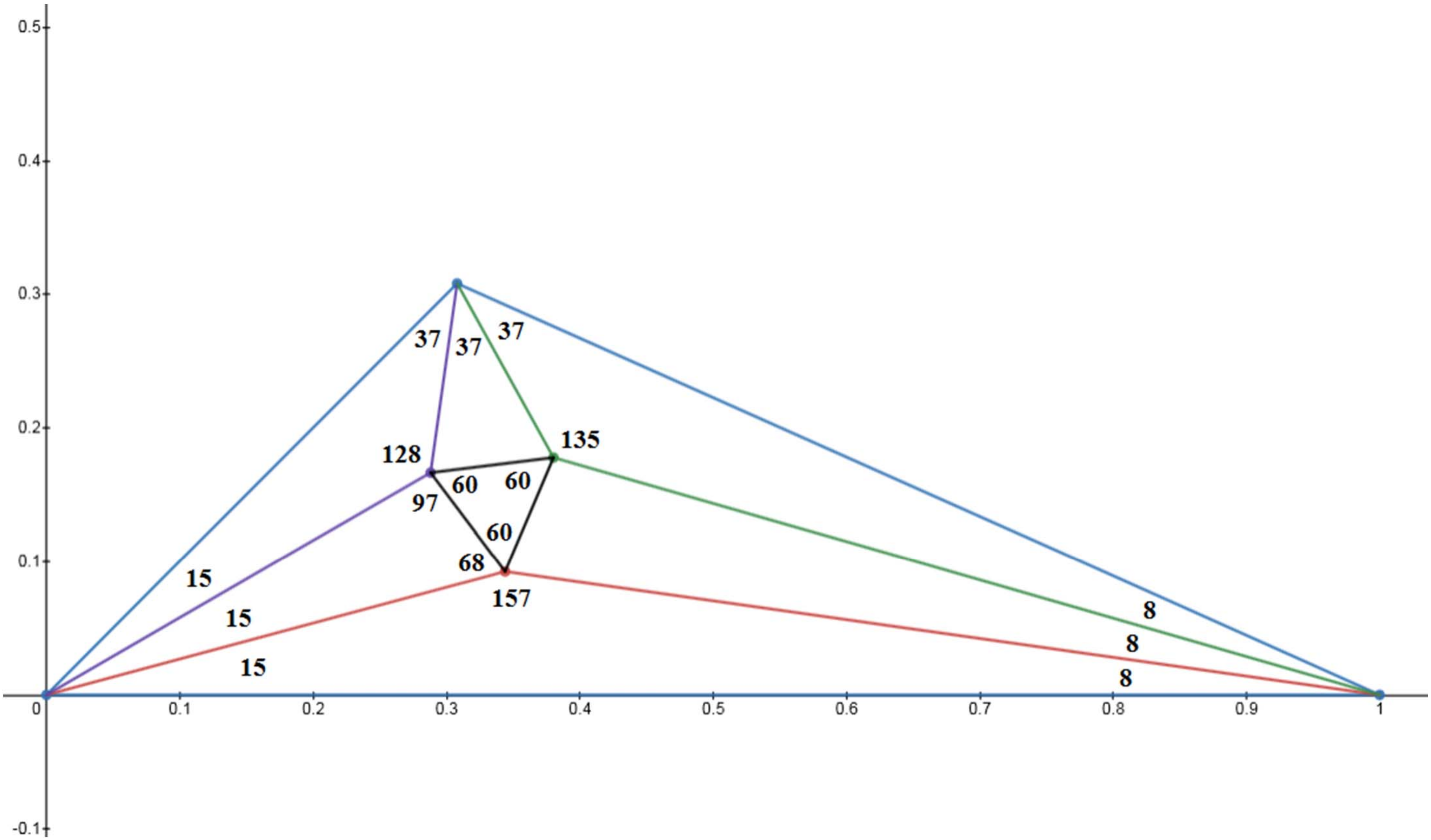
Earlier Example



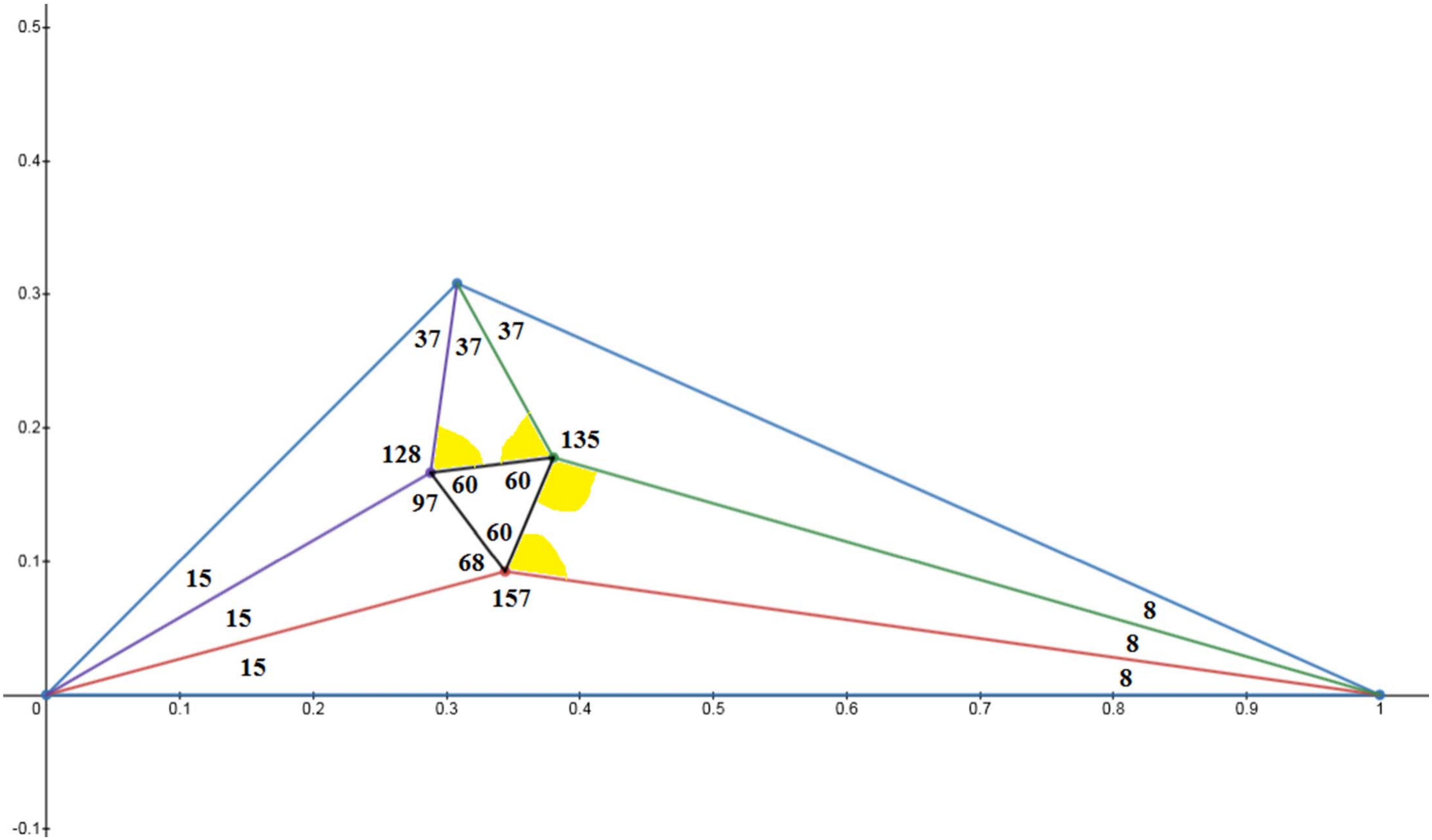
Earlier Example



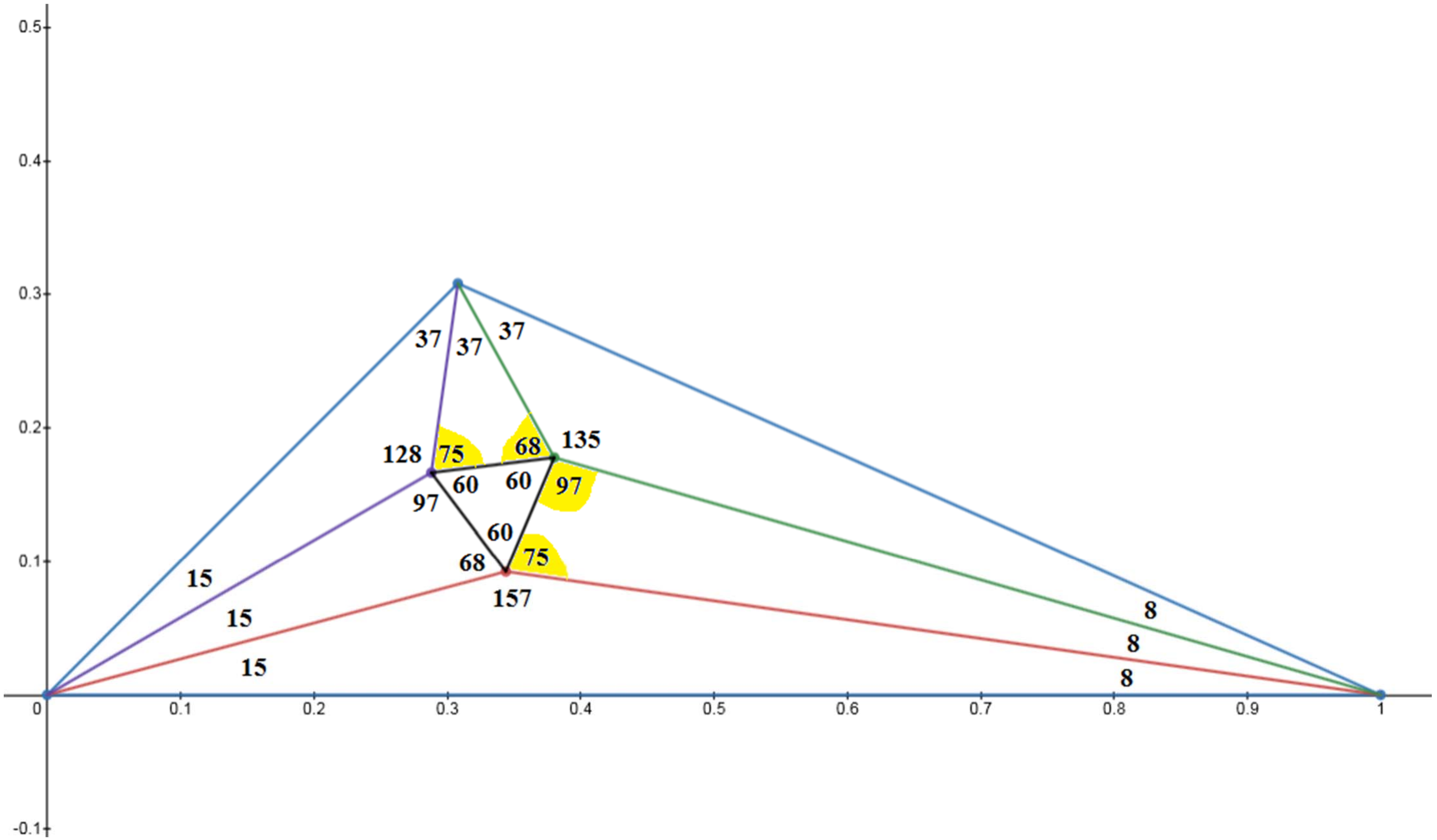
Earlier Example



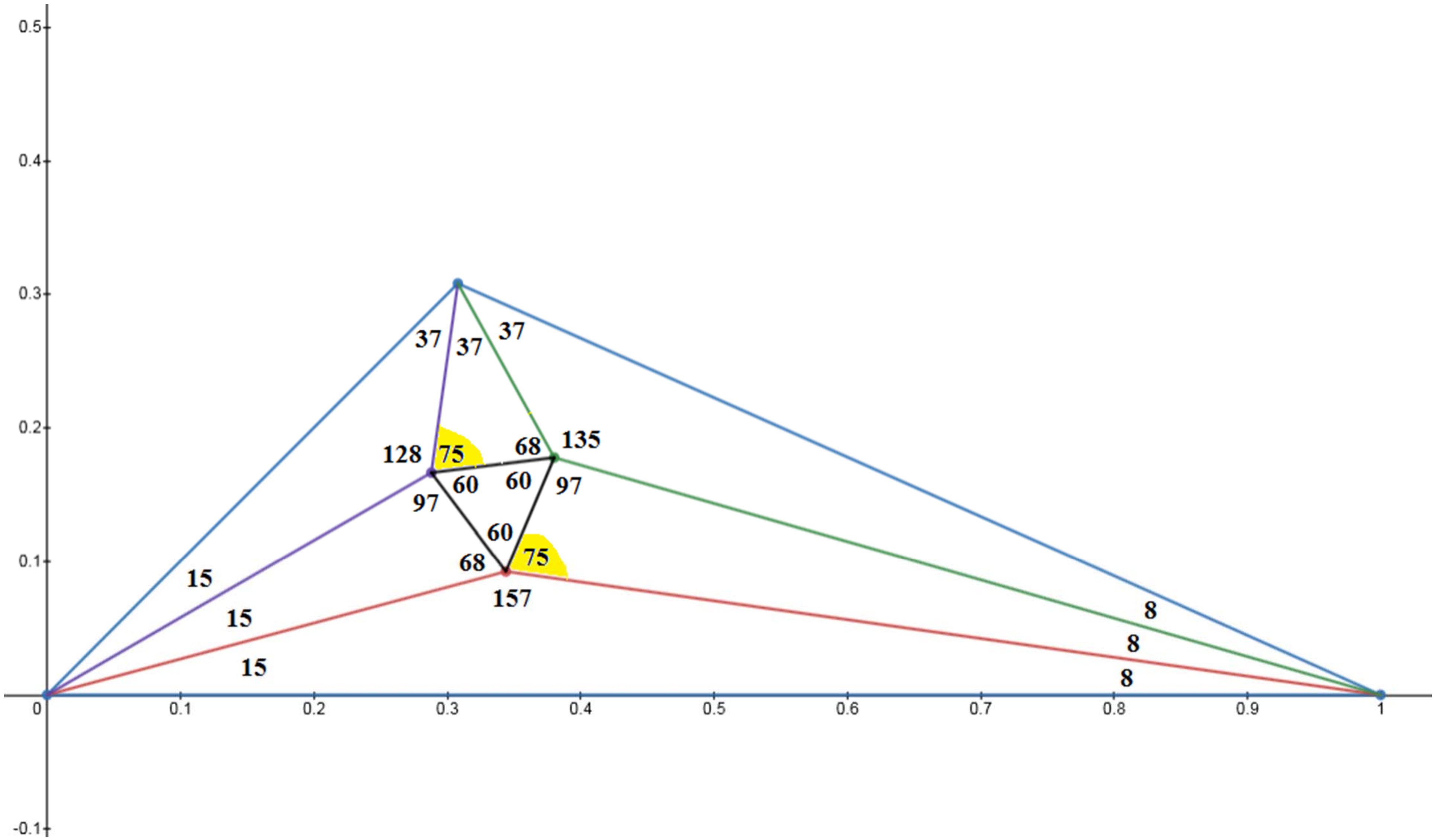
Earlier Example



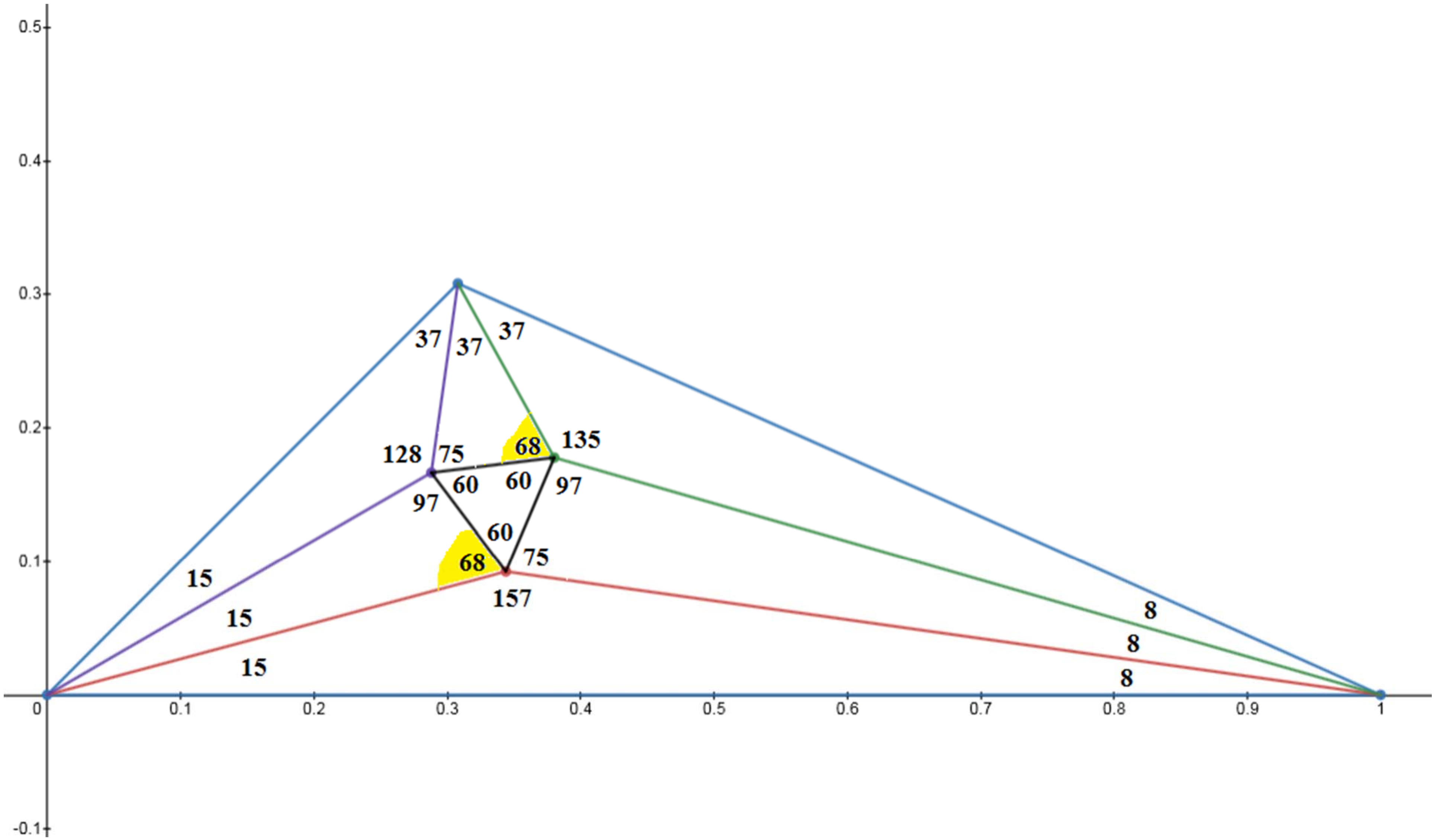
Earlier Example



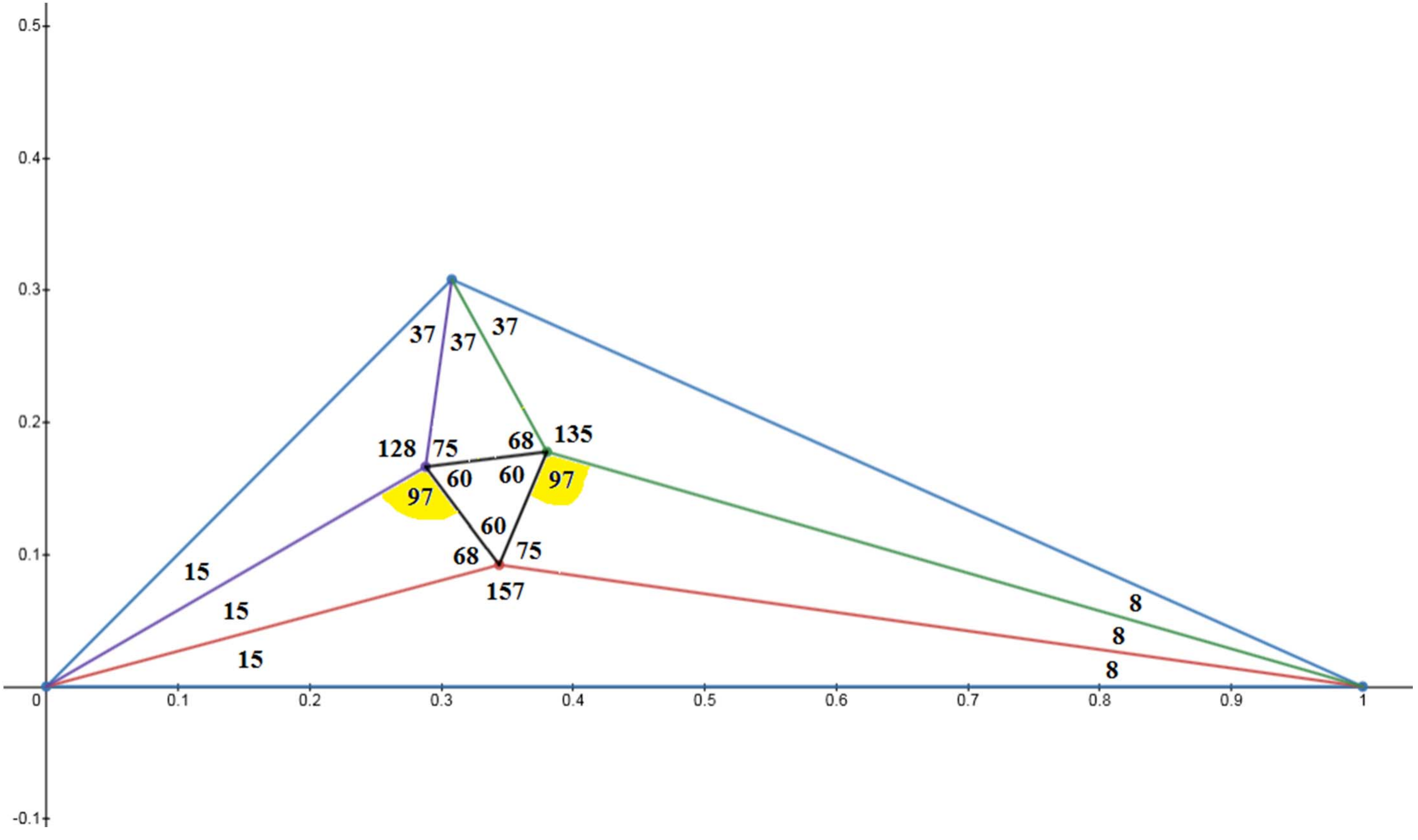
Earlier Example



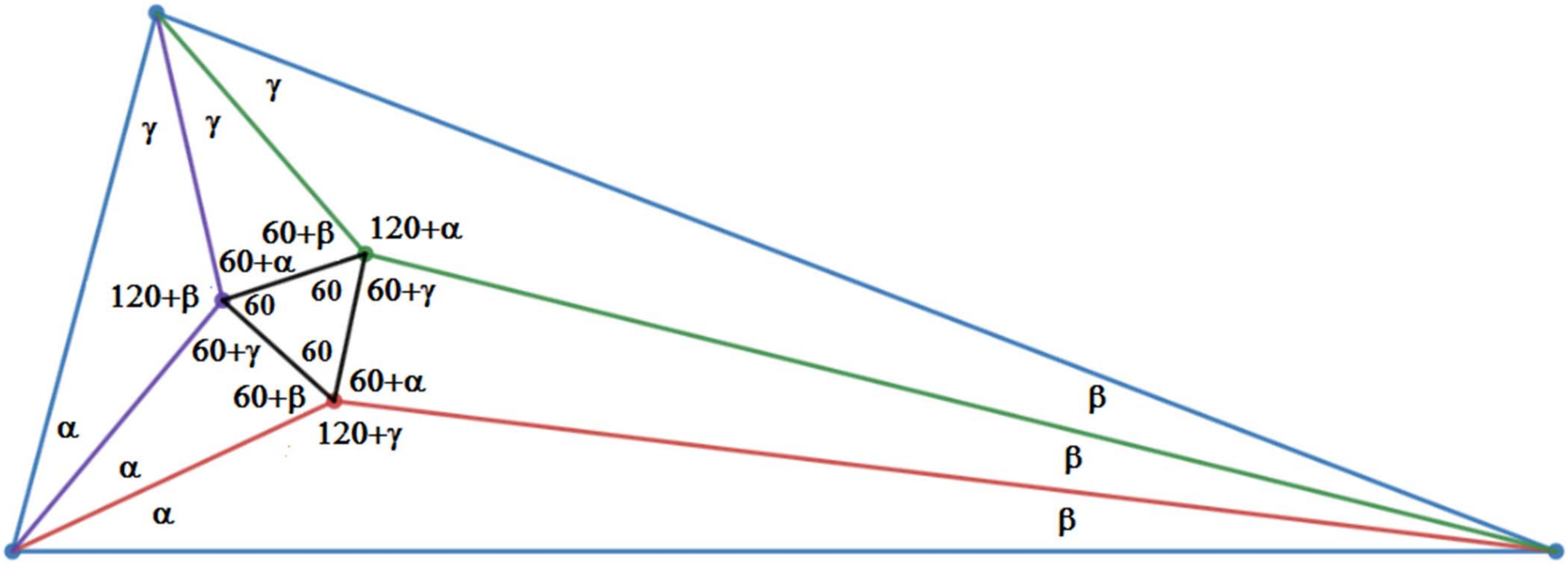
Earlier Example



Earlier Example

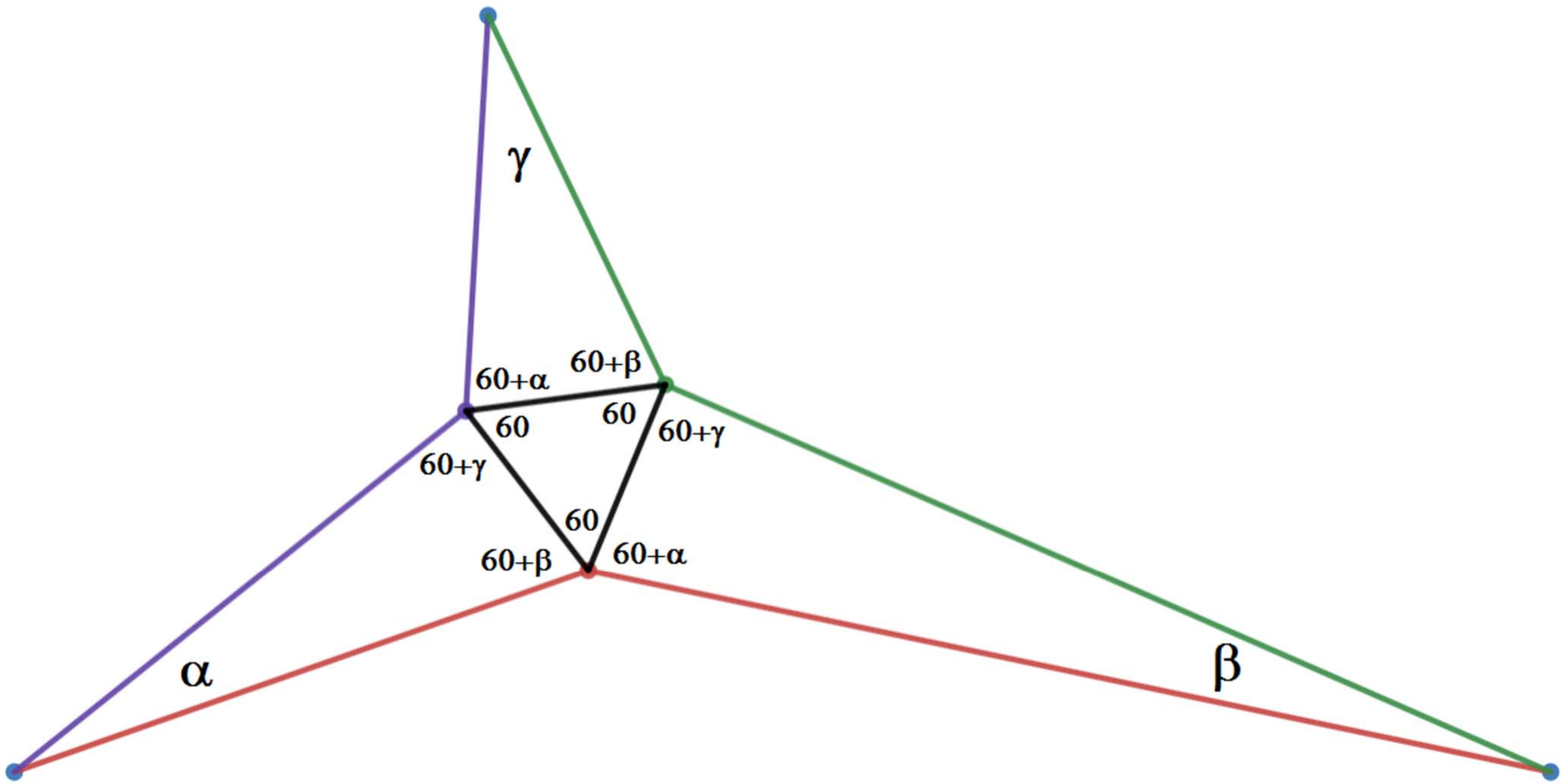


Obvious Conjecture

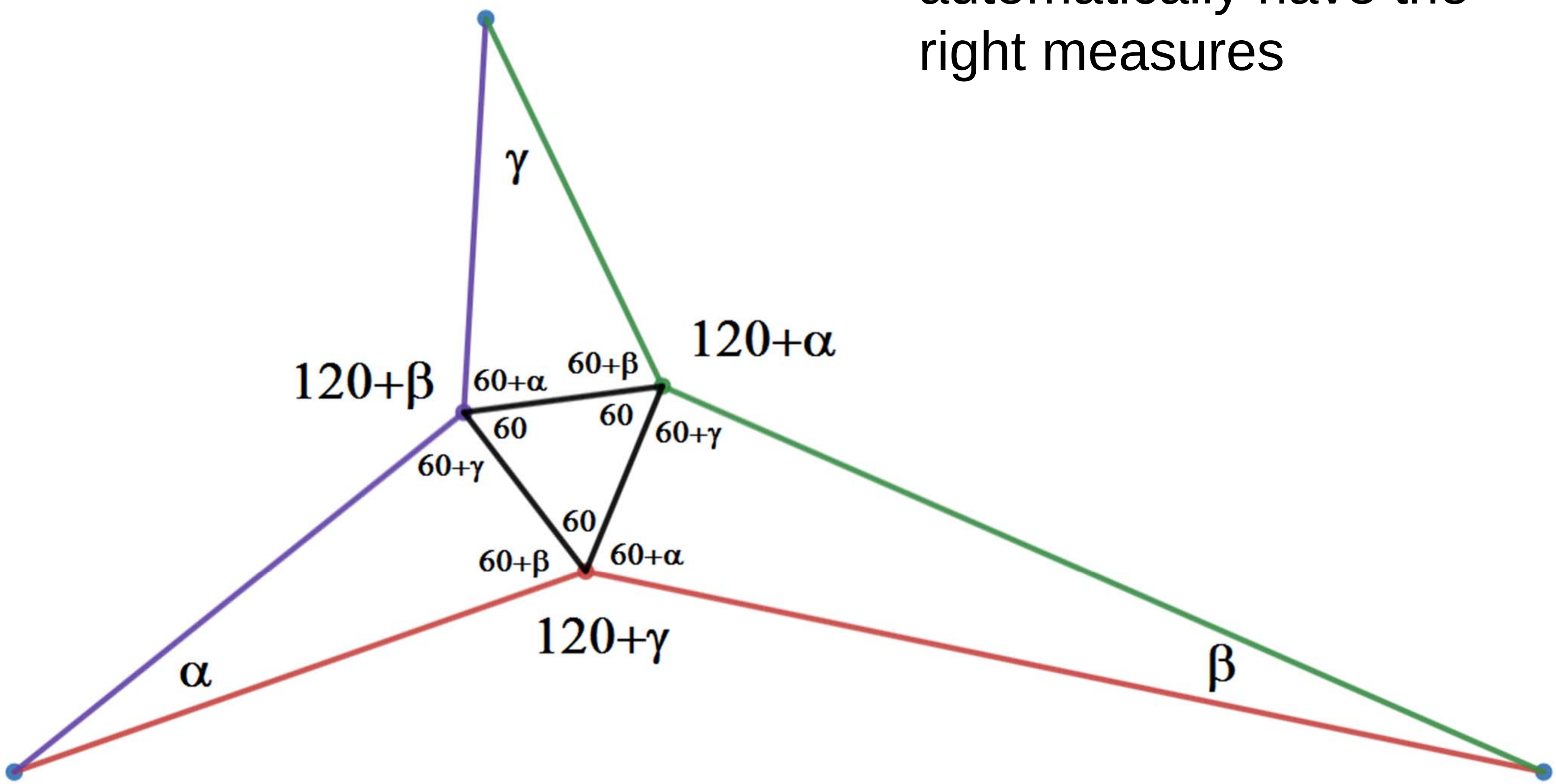


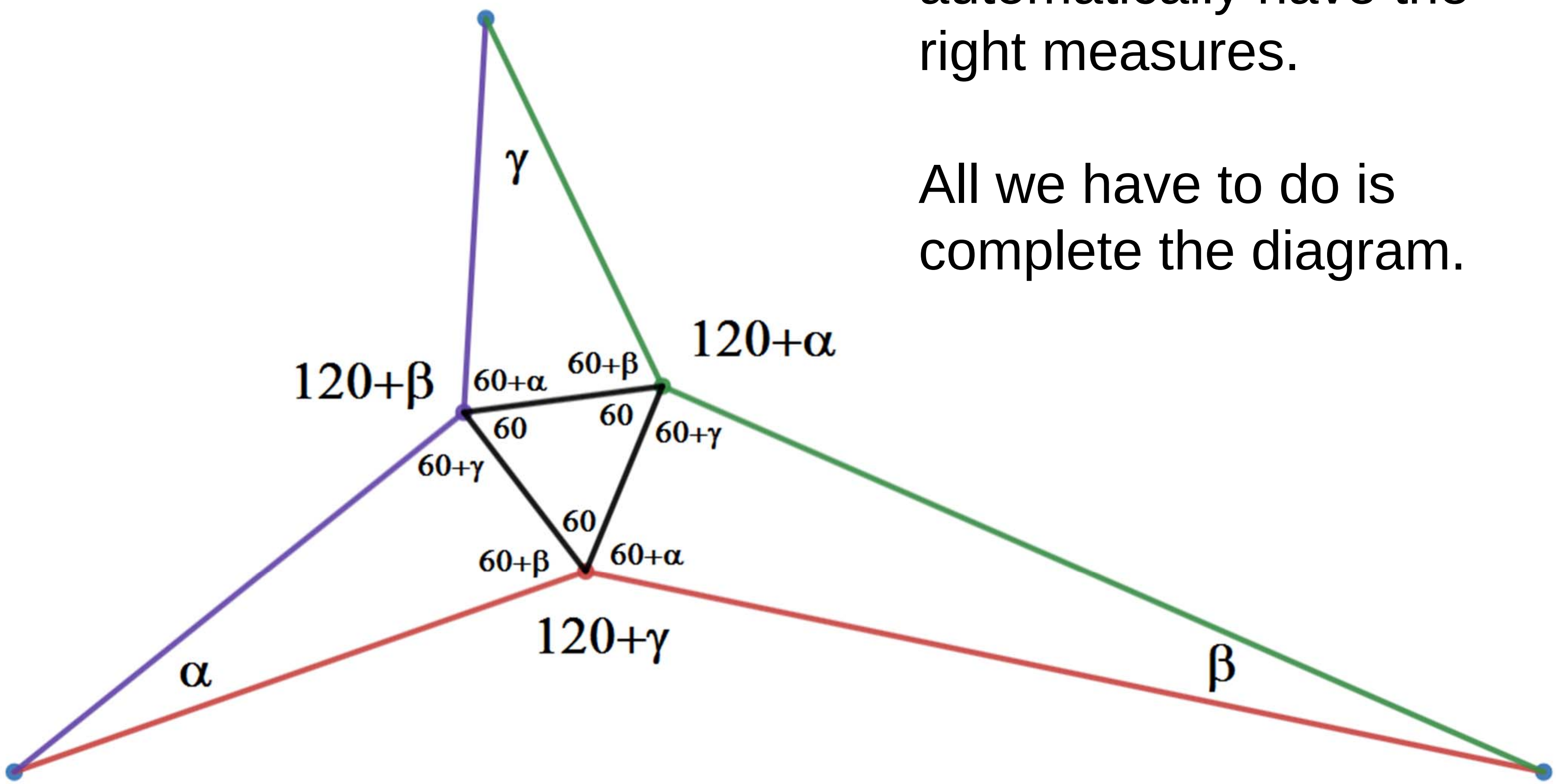
Reverse Engineered Proof

- Given a triangle, it is sufficient to prove the theorem for a similar triangle
- Thus, want to show for any triangle there is a similar triangle for which the conclusion of Morley's theorem holds
- So let the angles be given, and construct a figure like the one in the conjecture.
- Start with an equilateral triangle with sides = 1
- On each side construct a triangle with conjectured angles.


γ
60+α
60+β
60
60
60+γ
60+γ
60
60+β
60+α
α
β

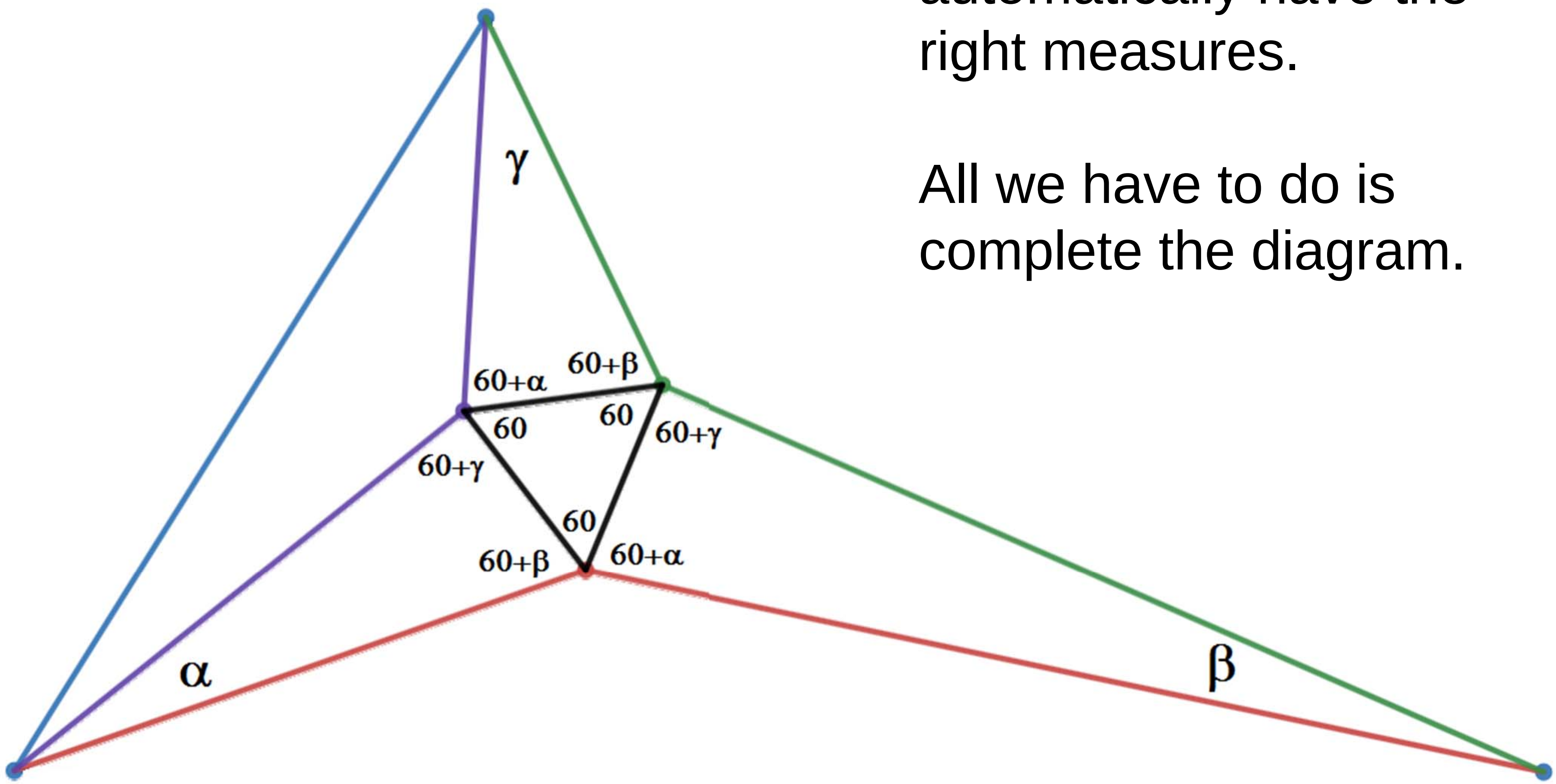
Exterior angles automatically have the right measures





Exterior angles automatically have the right measures.

All we have to do is complete the diagram.

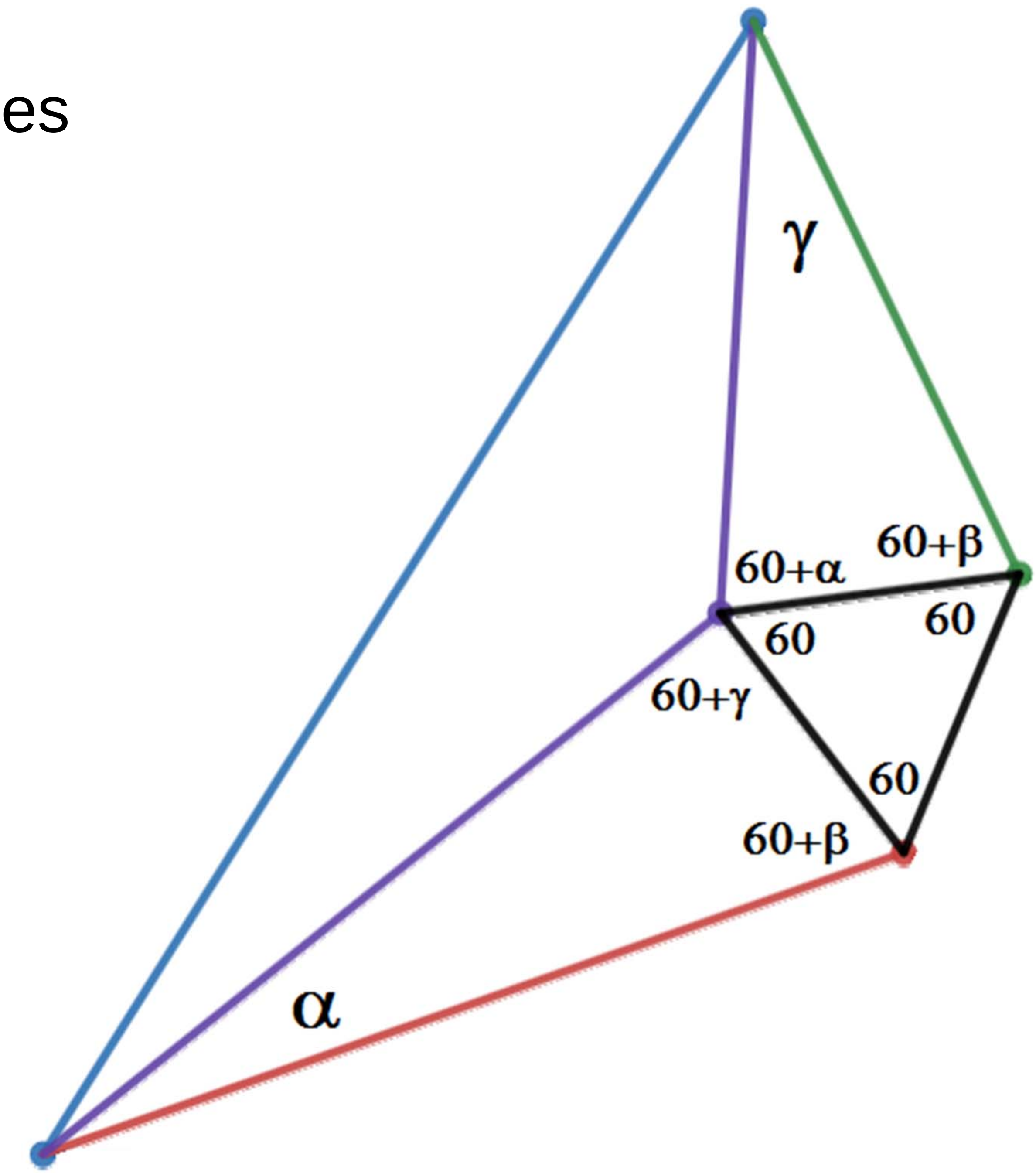


Exterior angles automatically have the right measures.

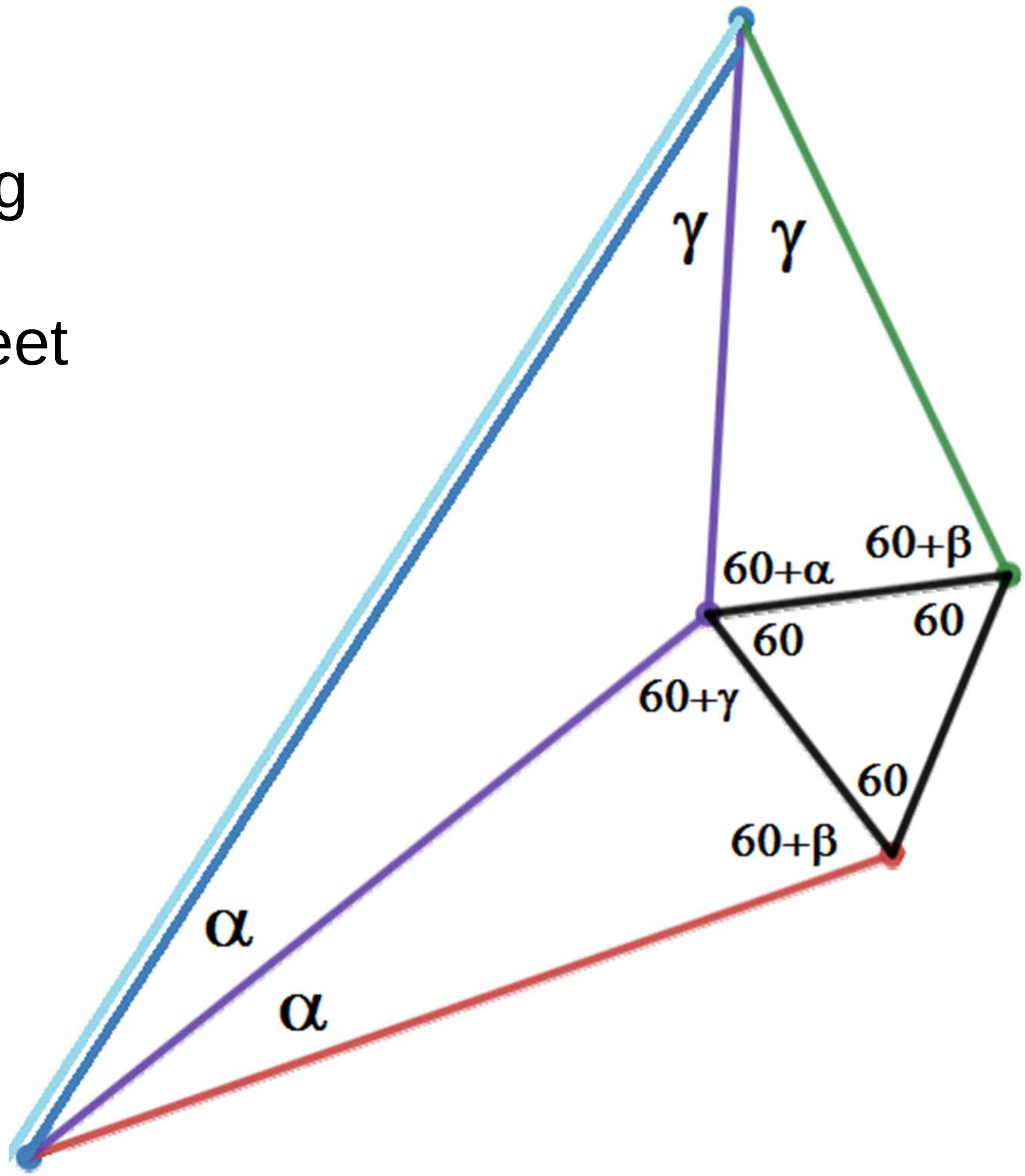
All we have to do is complete the diagram.

Focus on one additional line, at left.

- Connect the vertices
- Angles might not equal α and γ



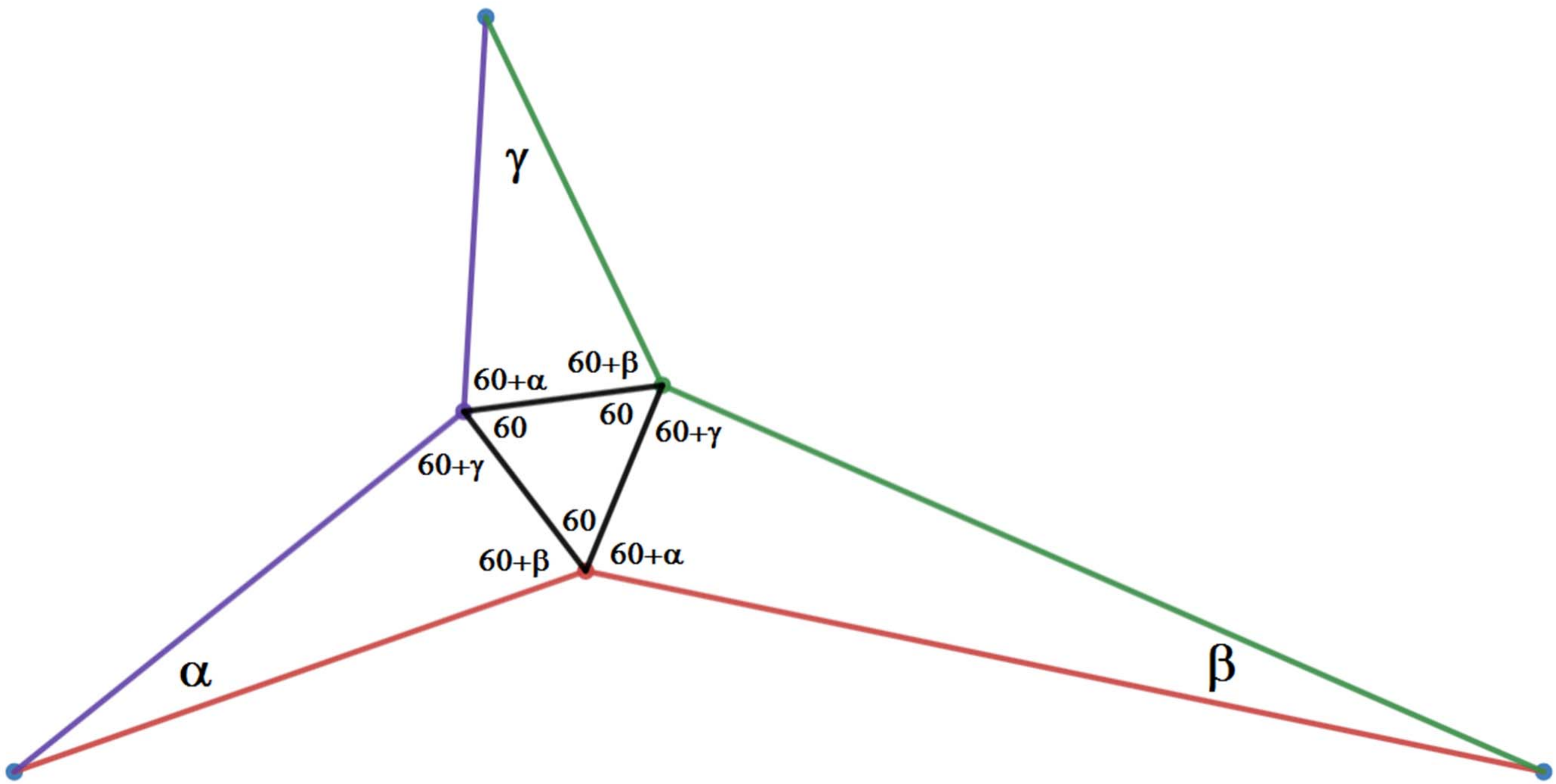
- Construct lines at each vertex making angles of α and β
- Lines might not meet up
- I'm stuck!



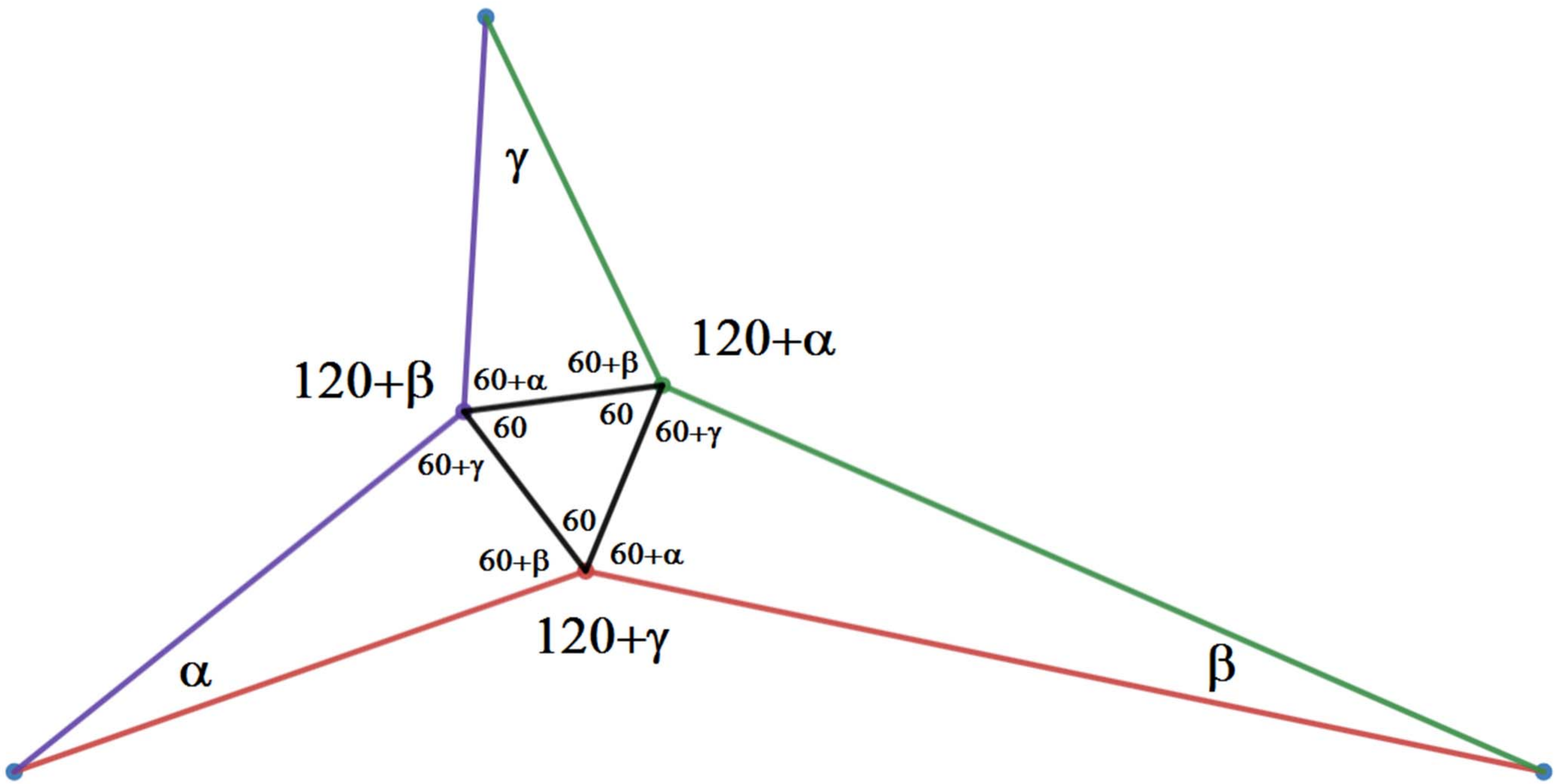
Conway's Proof

- Dunham told me my efforts were closely related to *famous* proof by Conway
- I found it on the internet: http://www.cut-the-knot.org/triangle/Morley/conway.shtml
- It was actually the same approach I had tried
- I got *this* close to finding a proof that Conway thought of!!!!
- Of course, *he* got all the way there!

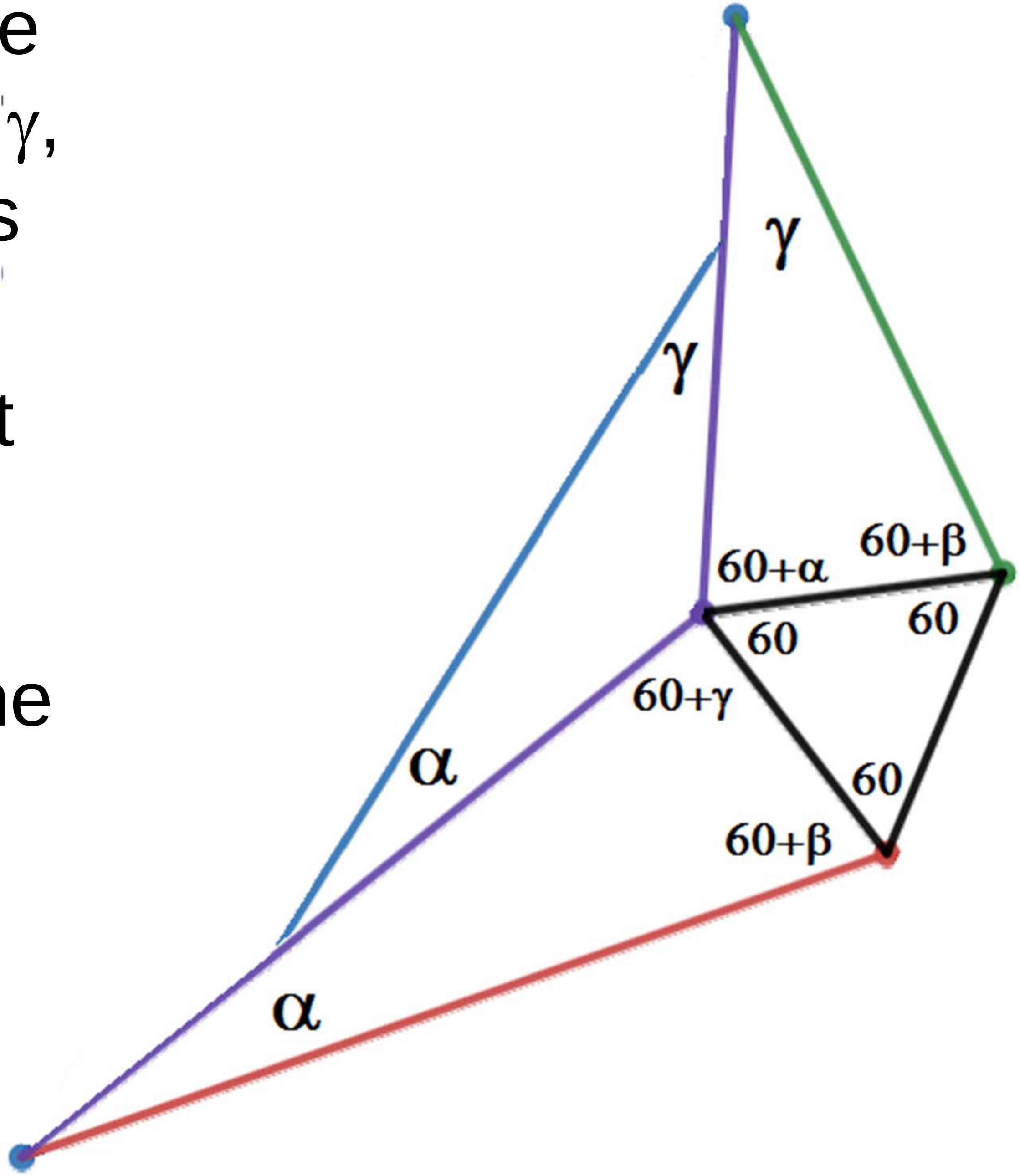
- Start with equilateral triangle, side = 1
- Construct triangles on sides as shown



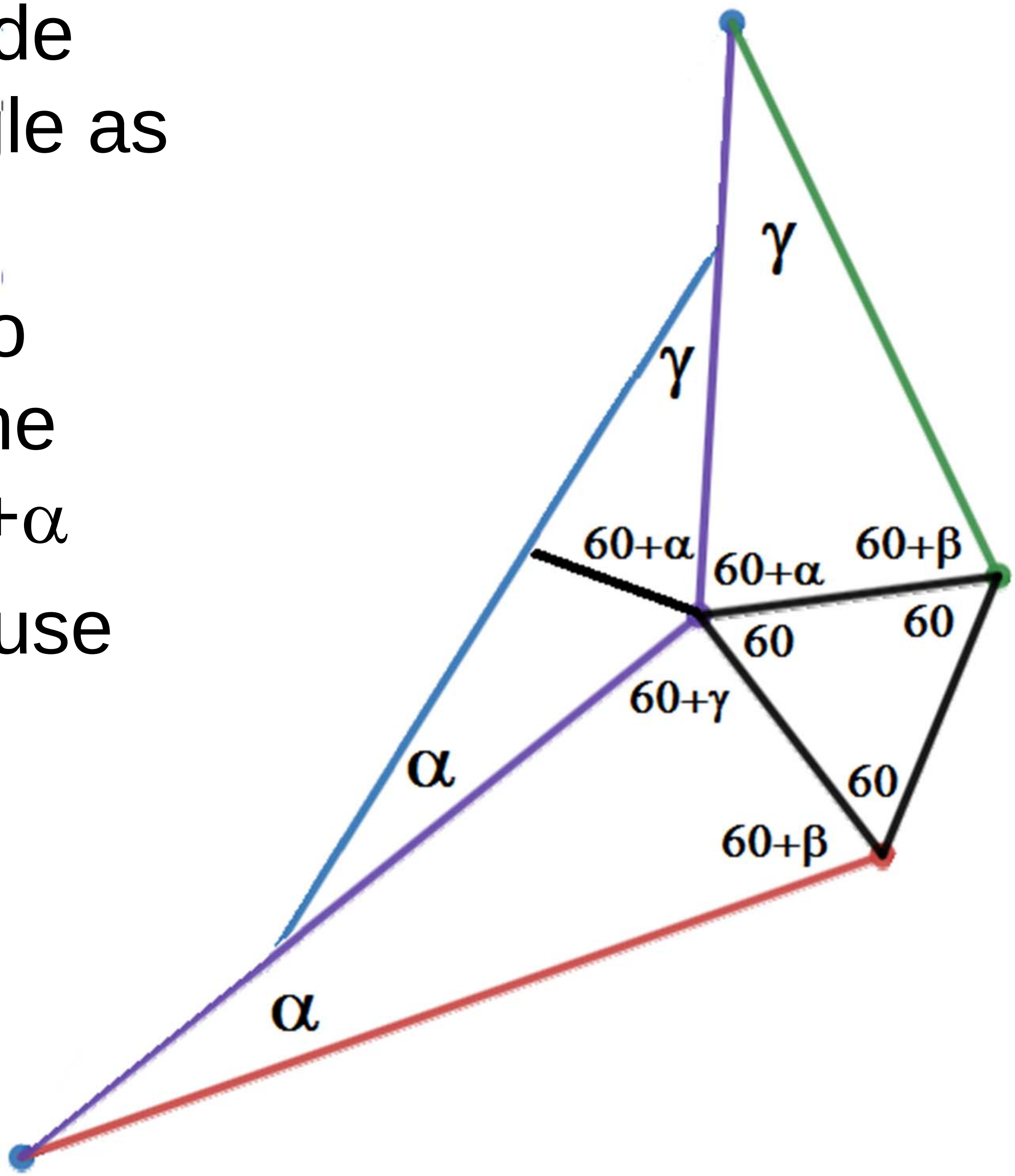
- 3 exterior angles found using 360° rule
- So far, all angles agree with conjectured arrangement



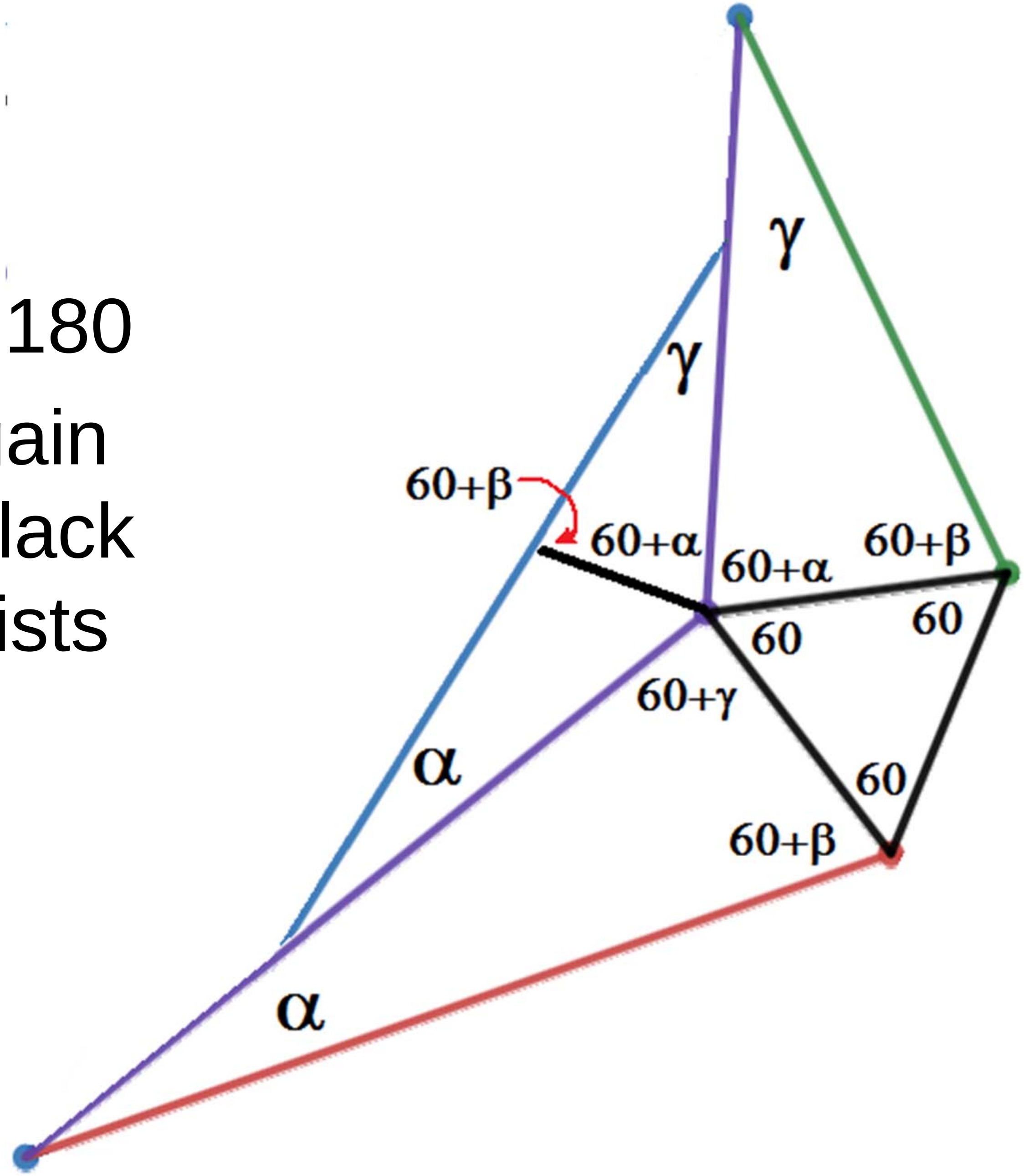
- Place a triangle with angles α, γ, and 120 + β as shown
- Scale it so that vertices lie on the purple segments in the figure.



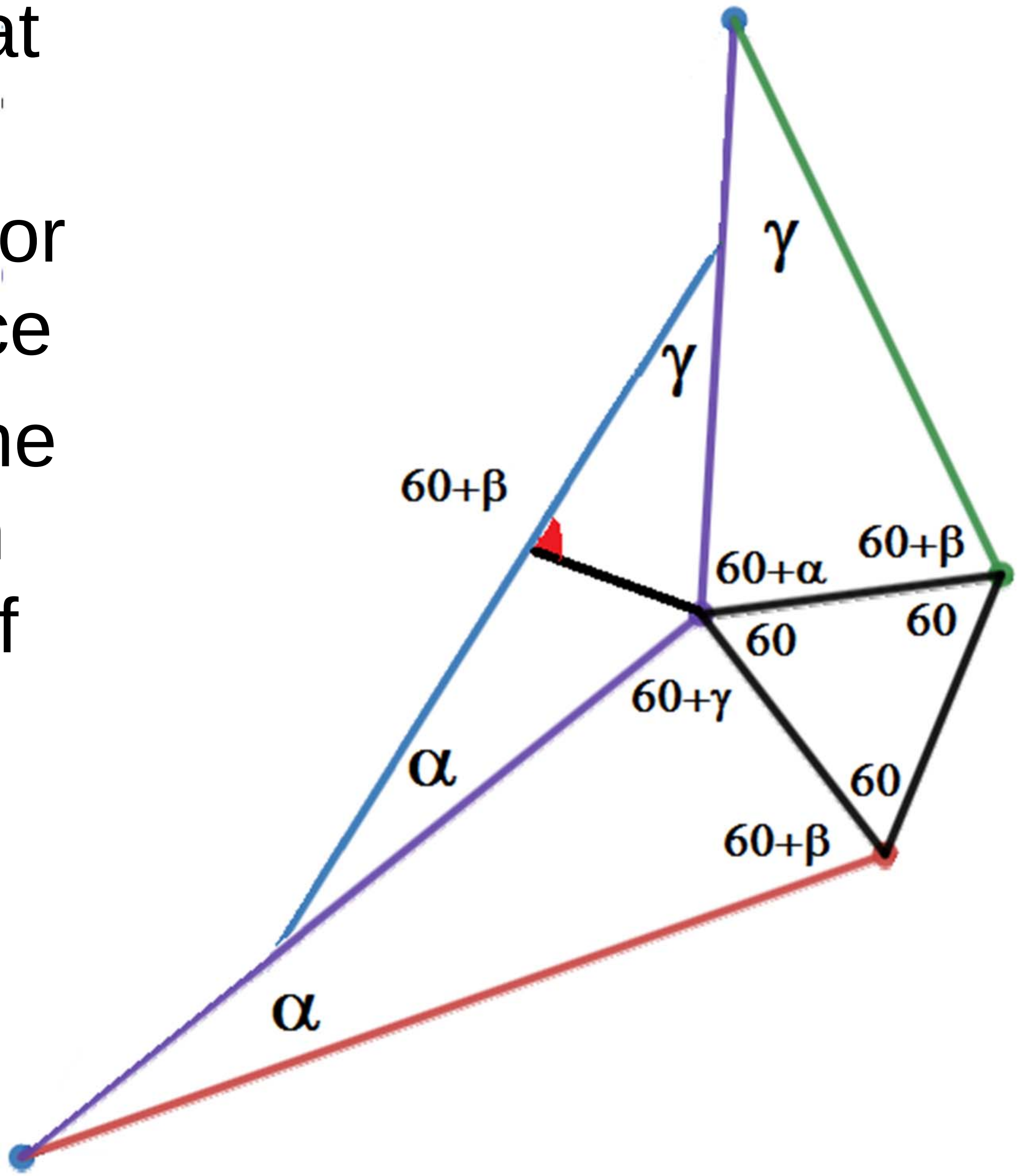
- Add a line inside this new triangle as shown
- Constructed so angle above the new line is 60+α
- Possible because
  $\alpha < 60 \Rightarrow$
  $60+\alpha < 120+\beta$



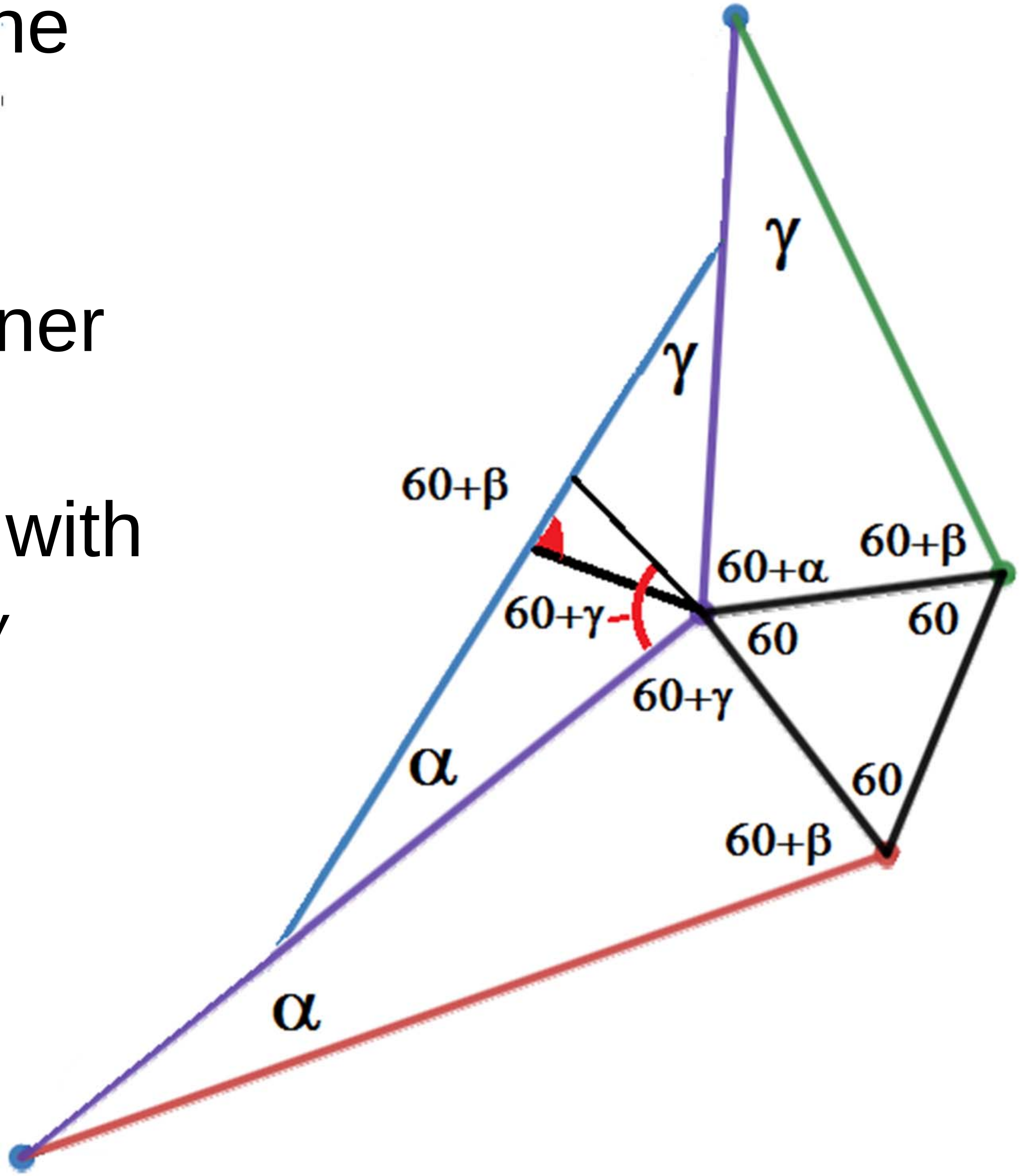
- Note that third angle is 60+β
- 60+60+α+β+γ = 180
- This shows again that the new black line always exists as shown



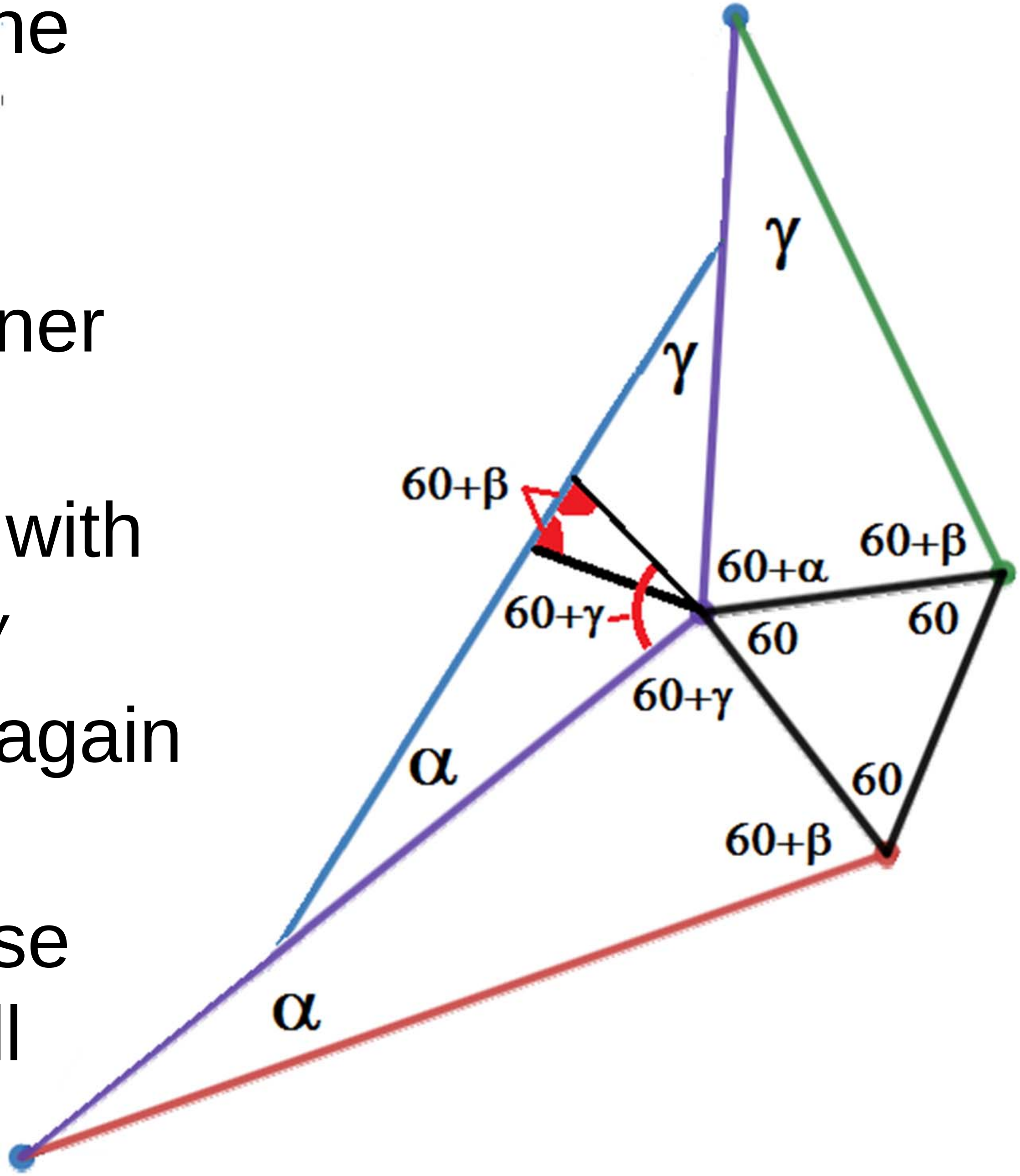
- Remember that angle: 60 + β
- Mark it in red for future reference
- Next: repeat the construction in the bottom half of the triangle with blue side



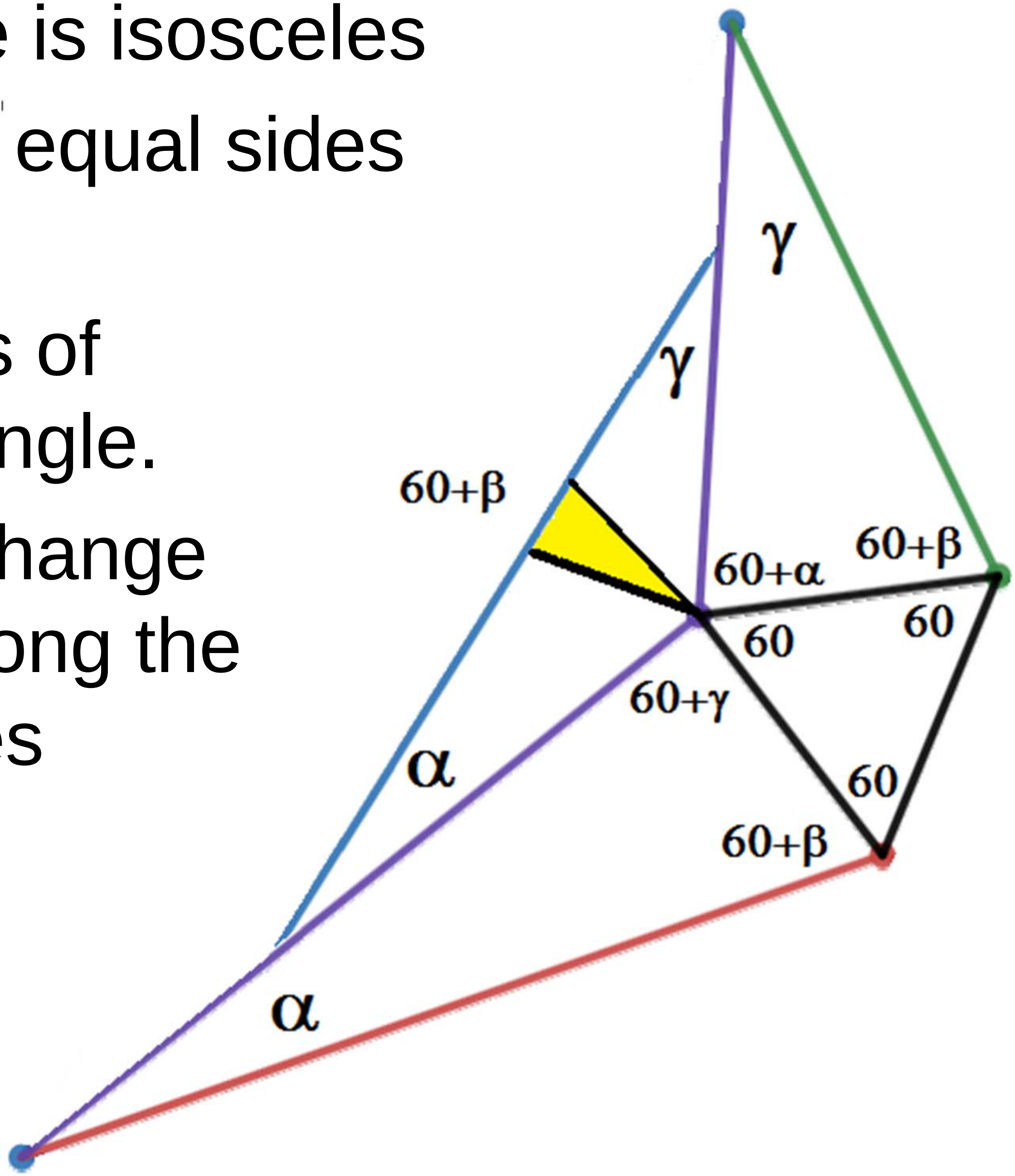
- Add another line above the one added before
- Shown as thinner black line
- Angle marked with red arc is 60+γ



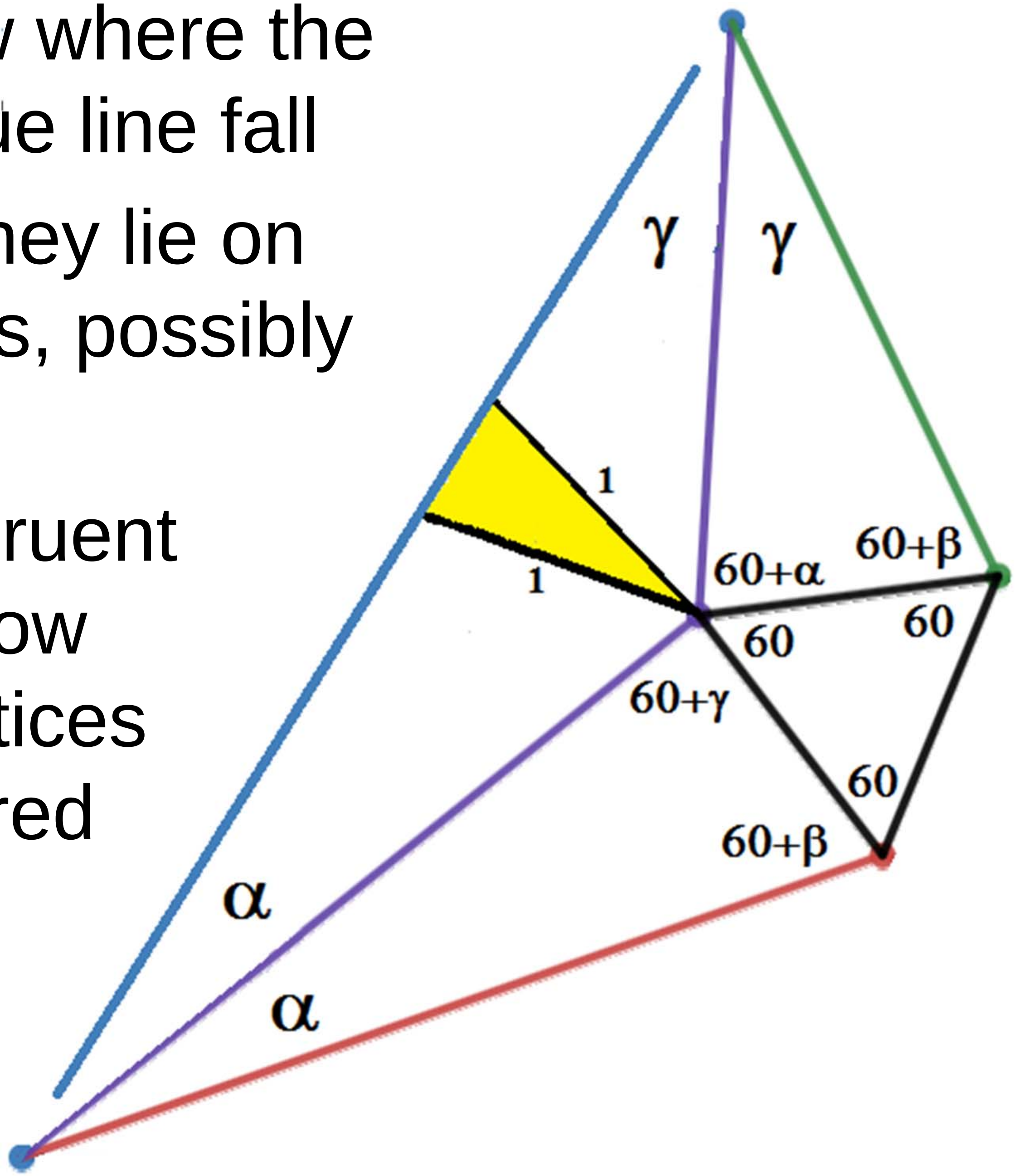
- Add another line above the one added before
- Shown as thinner black line
- Angle marked with red arc is 60+γ
- Third angle is again 60+β
- Note equal base angles of small triangle.



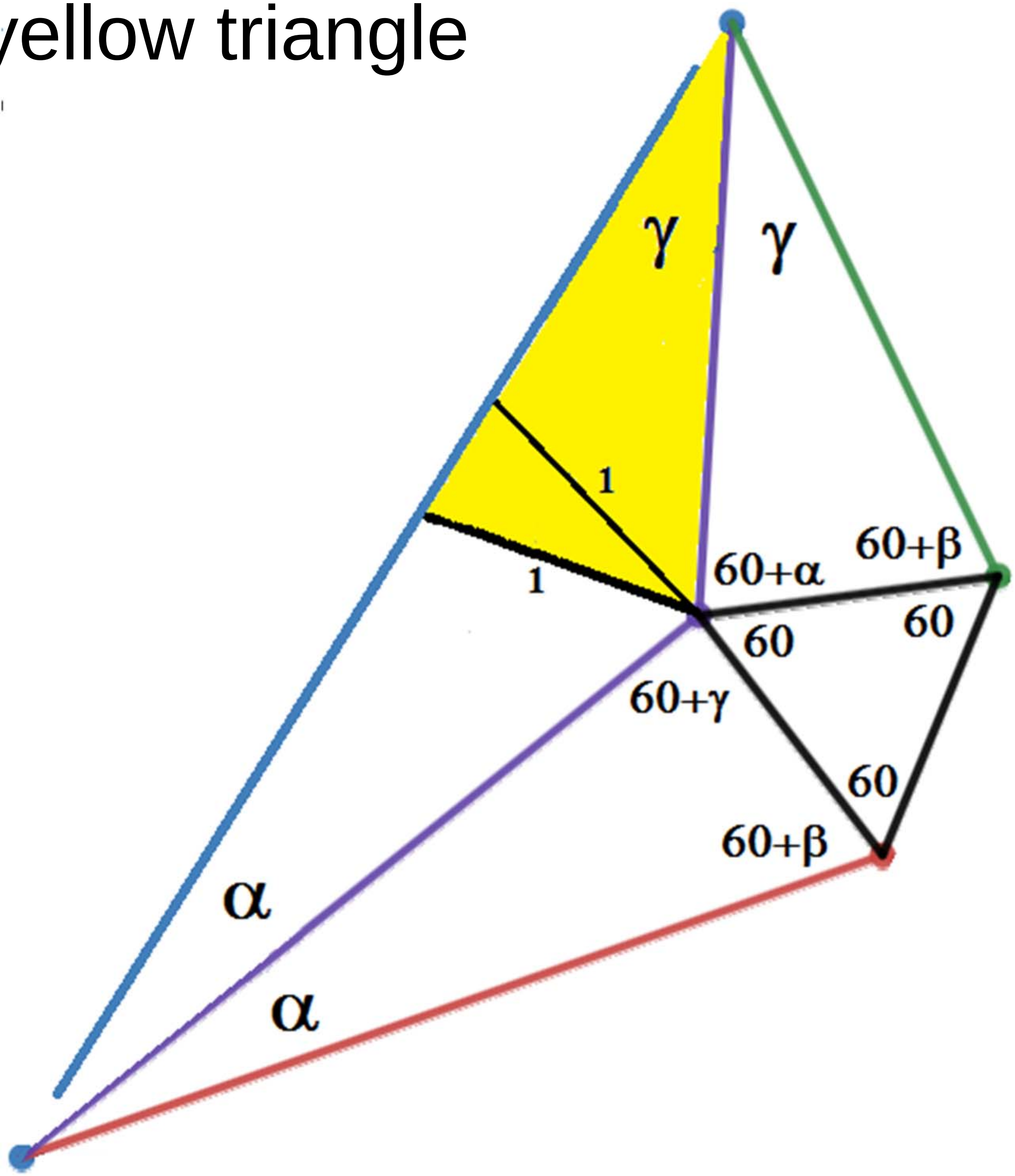
- Yellow triangle is isosceles
- Scale it so the equal sides have length 1
- Same as sides of equilateral triangle.
- Angles don't change so it still fits along the two purple lines



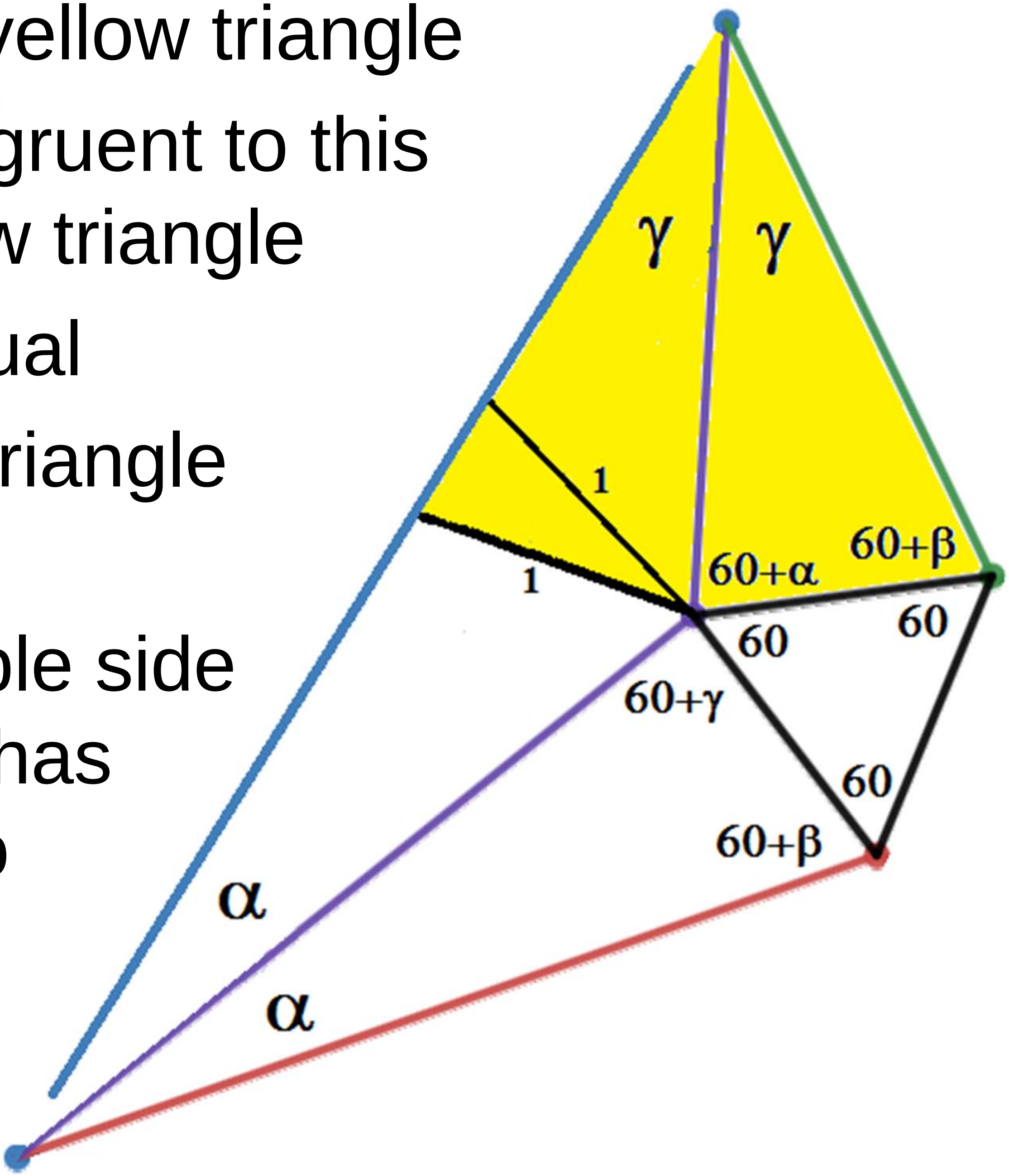
- Don’t yet know where the ends of the blue line fall
- We do know they lie on the purple lines, possibly extended
- Next use congruent triangles to show they fall at vertices on green and red sides.



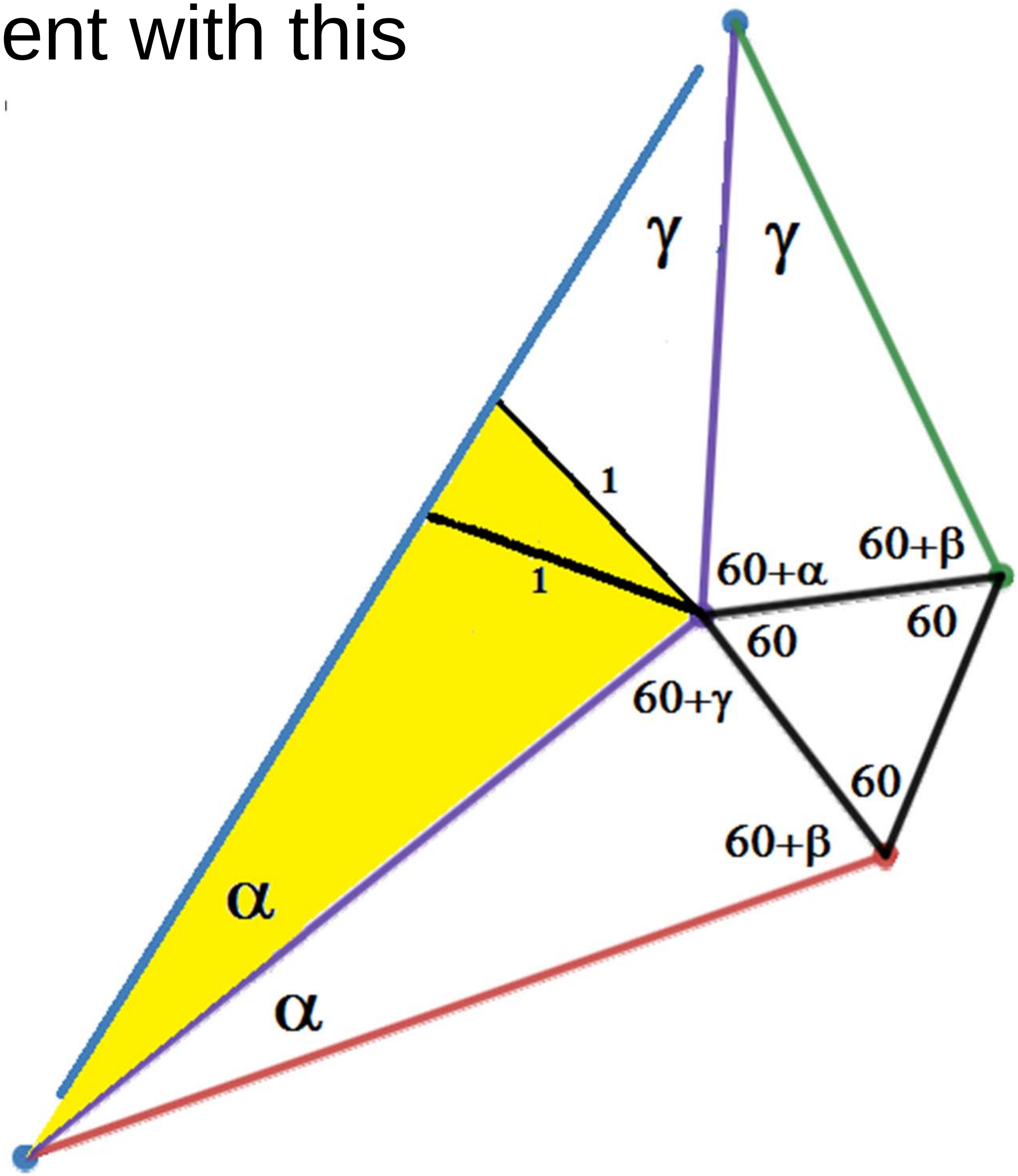
- Consider this yellow triangle



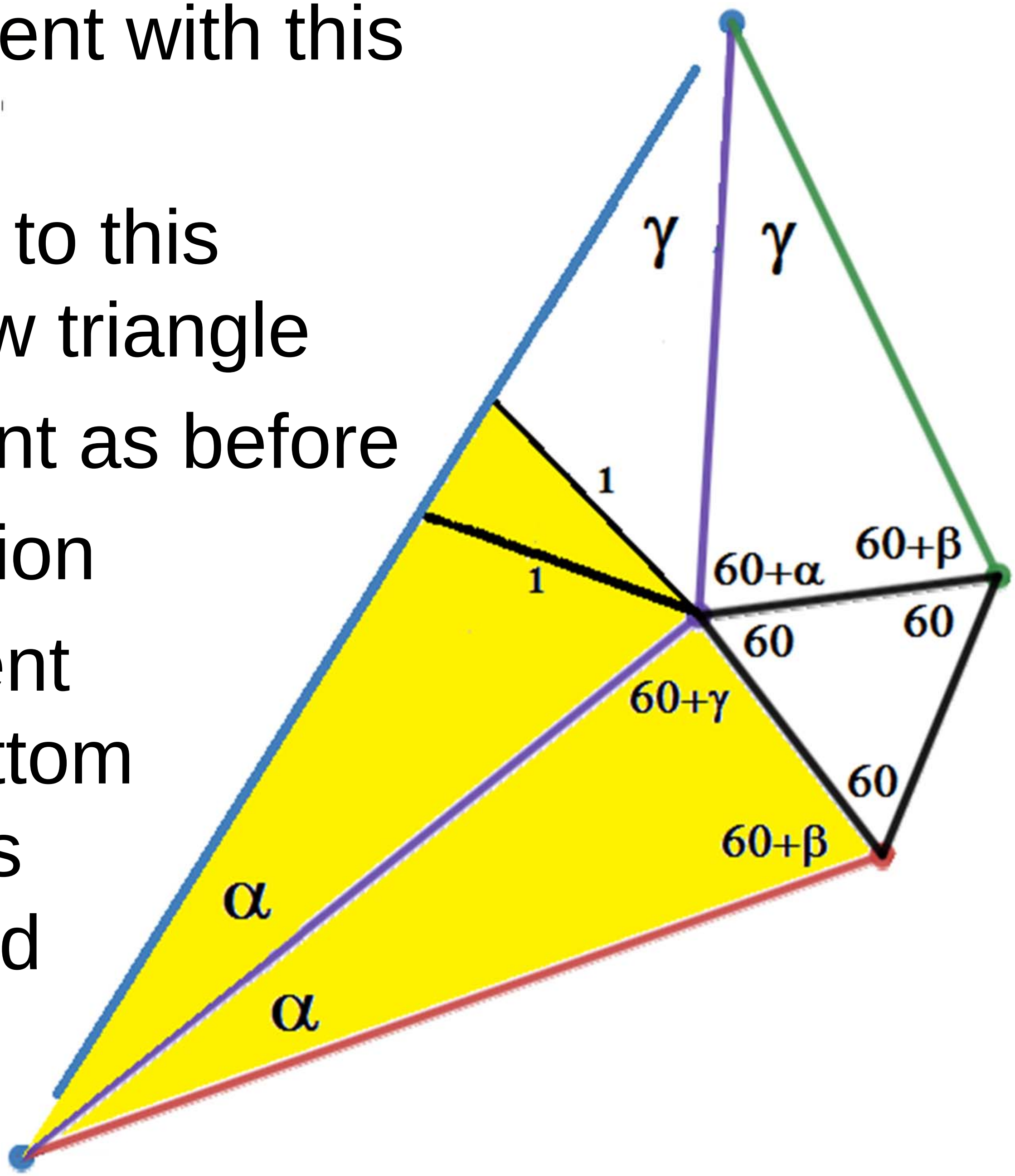
- Consider this yellow triangle
- Claim it is congruent to this adjacent yellow triangle
- Angles are equal
- Base of each triangle has length 1
- Therefore purple side of left triangle has equal length to purple side of right one.



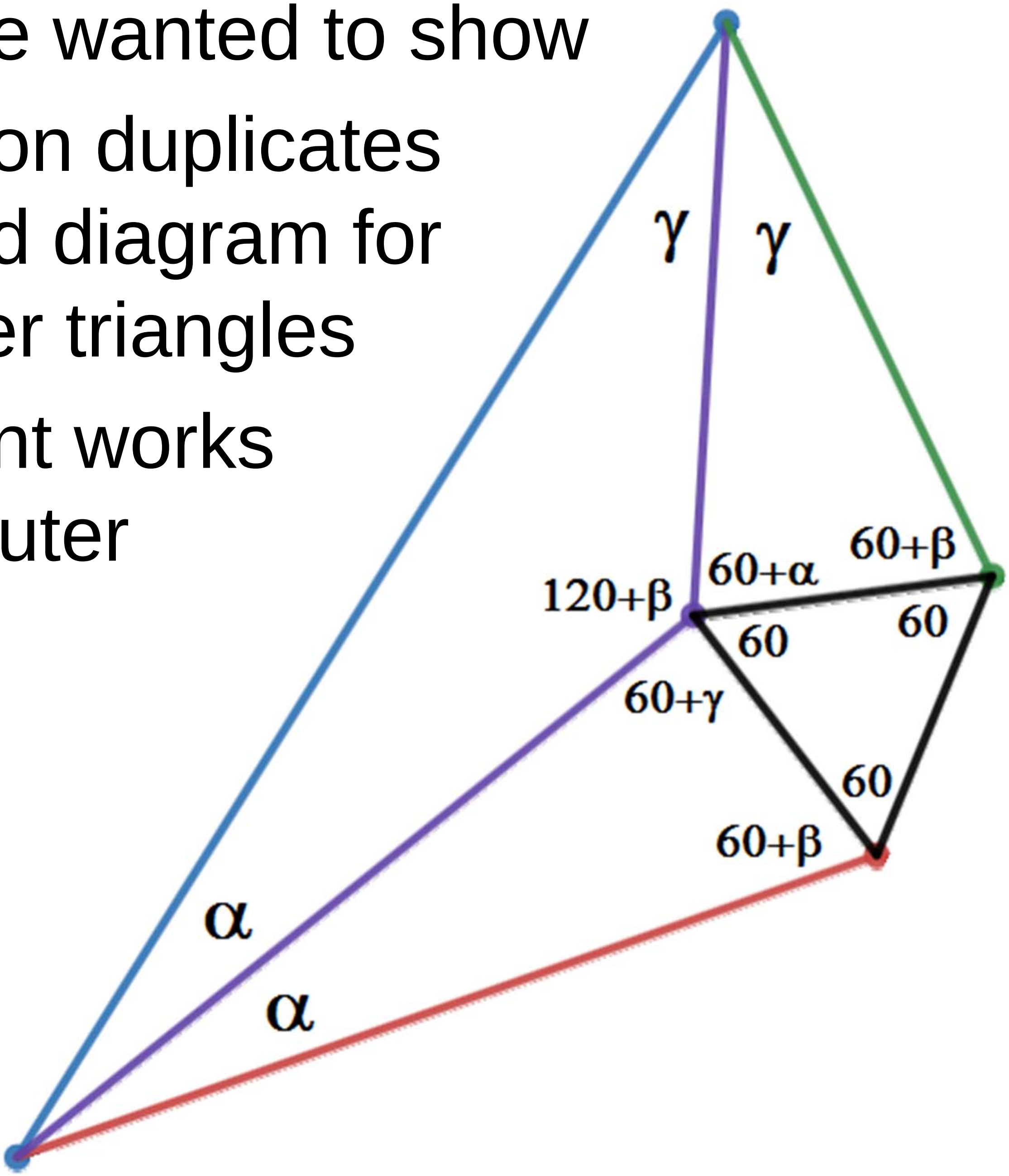
- Repeat argument with this yellow triangle



- Repeat argument with this yellow triangle
- It is congruent to this adjacent yellow triangle
- Same argument as before
- Same conclusion
- So: the segment from top to bottom vertices makes angles of α and γ with purple lines

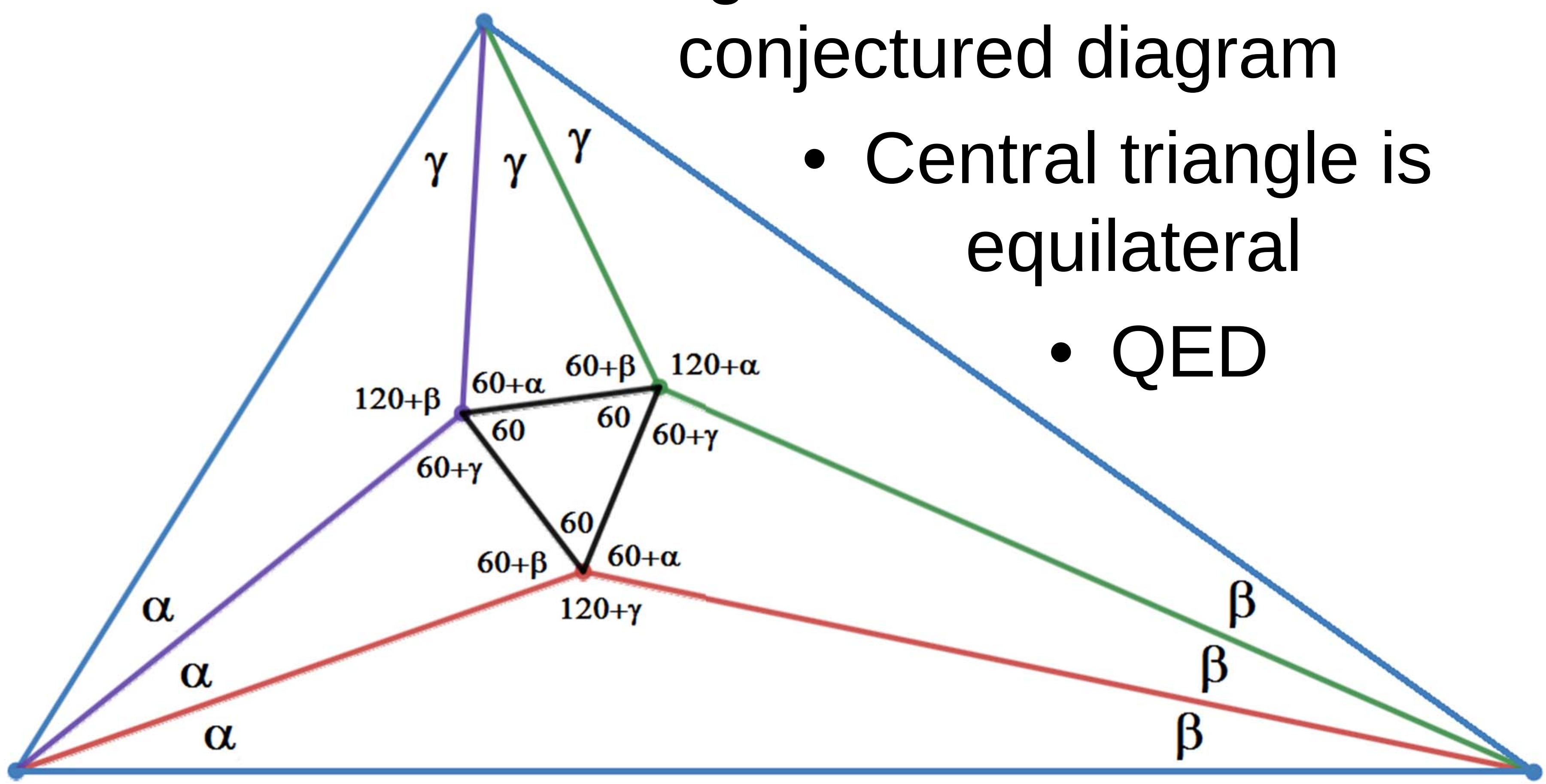


- This is what we wanted to show
- The construction duplicates the conjectured diagram for one of the outer triangles
- Same argument works for other two outer triangles.



Proof Complete

- Constructed a triangle with the given angles 3α, 3β, and 3γ
  - Angle trisectors create our conjectured diagram
    - Central triangle is equilateral
      - QED



Final Comments

- There are three cases of the Conway construction that have to be considered
- Case we showed: $\beta < 30$
- Similar arguments apply if $\beta = 30$ or $\beta > 30$.
- Many other proofs have been proposed. Several of them are presented: http://www.cut-the-knot.org/triangle/Morley/conway.shtml
- Dunham's quest for a synthetic (*ie* not reverse engineered) angles-only proof remains unfulfilled