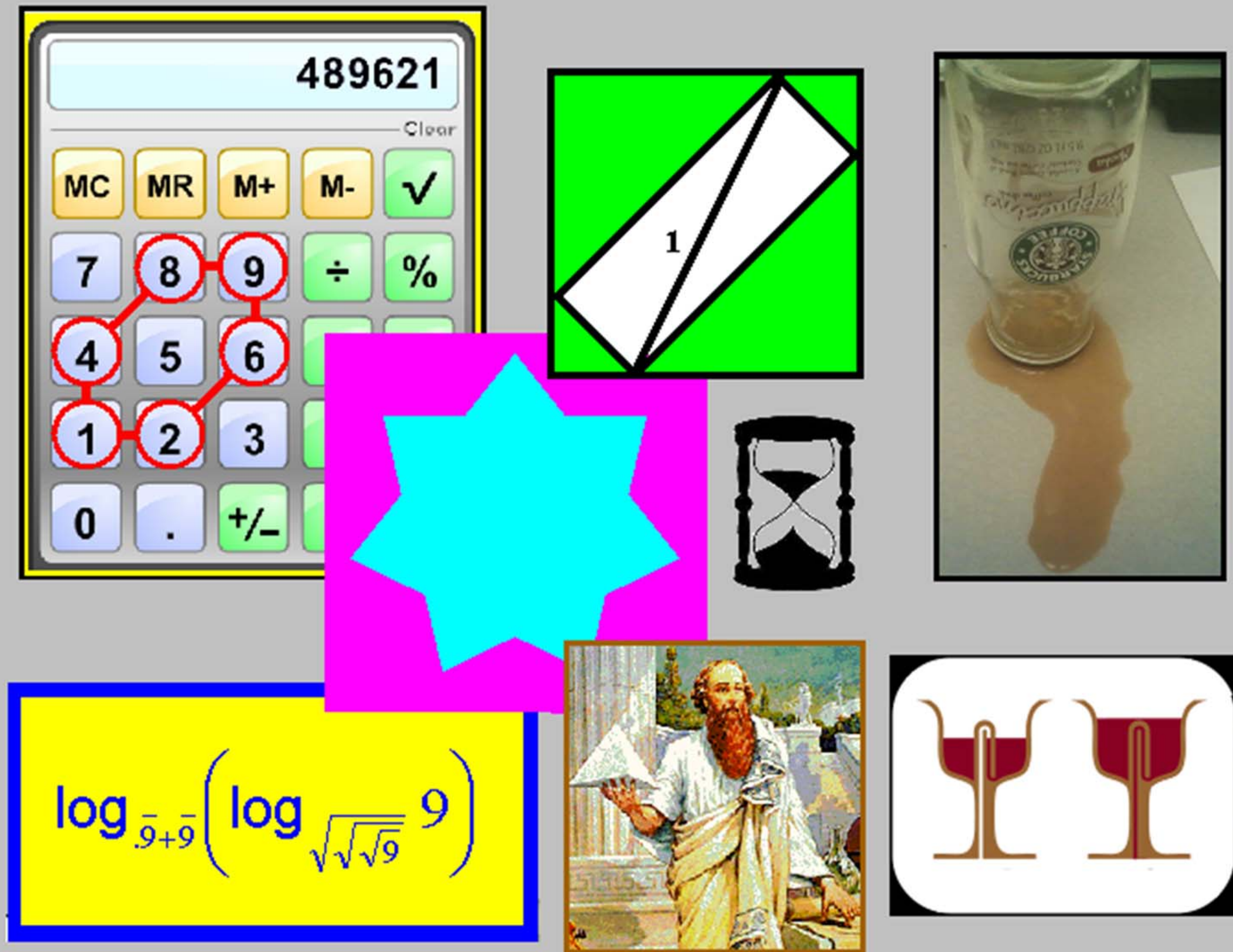
MINUTE MATH



Dan Kalman • American University

www.dankalman.net

Mental Magic

- Make up a polynomial $p(x)$:
  - –Positive integer coefficients
  - –All coefficients less than 10
  - –Any degree you choose
- I will guess your polynomial
- I need one hint
- What is $p(10)$?

More generally …

- Let $p(x)$ be a polynomial of any degree with non-negative integer coefficients
- You can find p exactly by knowing just two values: $p(1)$ and $p(p(1))$
- Let $b = p(1)$ (this is the sum of coefficients)
- Given $p(b)$, express it in base b notation.
- Read off the coefficients.

The
Quadratic Formula
Inside Out

- Let $p(x) = (x - r)(x - s)$
- Expand to find $p(x) = x^2 - (r + s)x + rs$
- Now quad formula gives the roots as

$$\frac{r+s\pm\sqrt{(r+s)^2-4rs}}{2}=\frac{r+s\pm\sqrt{(r-s)^2}}{2}$$

$$=\frac{r+s\pm(r-s)}{2}$$

- When ± is + we get the larger root
- Theorem: given numbers r and s the larger is r

- Let $p(x) = (x - r)(x - s)$
- Expand to find $p(x) = x^2 - (r + s)x + rs$
- Now quad formula gives the roots as

$$\frac{r+s\pm\sqrt{(r+s)^2-4rs}}{2}=\frac{r+s\pm\sqrt{(r-s)^2}}{2}$$

$$=\frac{r+s\pm|r-s|}{2}$$

- This should give the roots: r and s -- what gives?
- Formulas for max and min of two quantities

The
Four 9’s Puzzle

Express a number with four 9's

- **1 = 99/99**
- **2 = 9/9 + 9/9**
- **3 = (9+9+9)/9**
- **4 =** $\left(\sqrt{9}\right)! - \frac{9+9}{9}$
- **5 =** $\frac{9+9}{9} + \sqrt{9}$

Universal Solution

$$\log_{.\bar{9}+.\bar{9}}\left(\log_{\sqrt{\sqrt{\cdots\sqrt{9}}}} 9\right)$$

- With n nested radicals, 2nd log base is
  $b = 9\text{^}((1/2)^n) = 9\text{^}(2^{-n})$
- $9 = b\text{^}(2^n)$
- $\log_2 (\log_b (9)) = \log_2 (2^n) = n$

Author! Author!

- Terry Moore, AU masters student, 2000
- Solution based on an earlier creation of Verner Hoggatt
- He founded the Fibonacci Quarterly
- See: Howard Eves, "Hail to thee, blithe spirit!", Fibonacci Quarterly, 19 (1981) 193-196. http://www.fq.math.ca/Scanned/19-3/eves.pdf

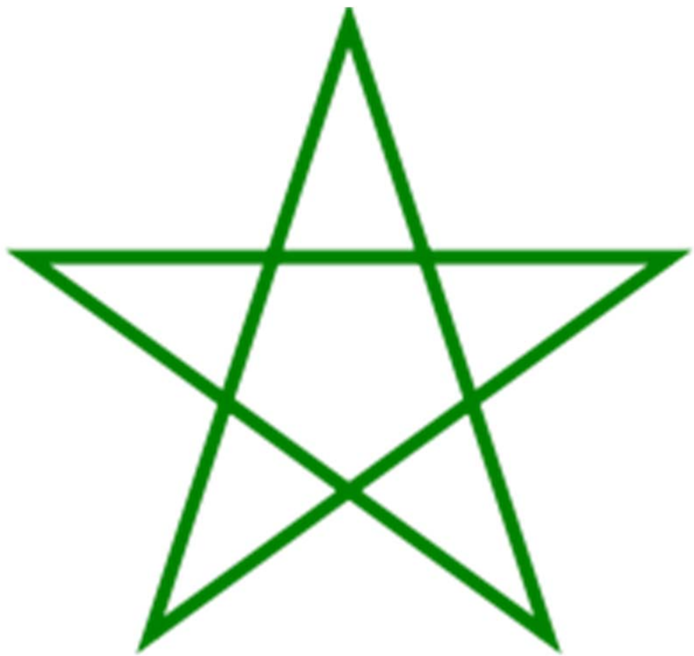
Hoggatt's Formula

- Expression for any positive integer *n*
- Each decimal digit appears once, in order
- Universal Solution

$$\log_{(0+1+2+3+4)/5}\left(\log_{\sqrt{\sqrt{\cdots\sqrt{-6+7+8}}}} 9\right)$$

A Stellar Coincidence

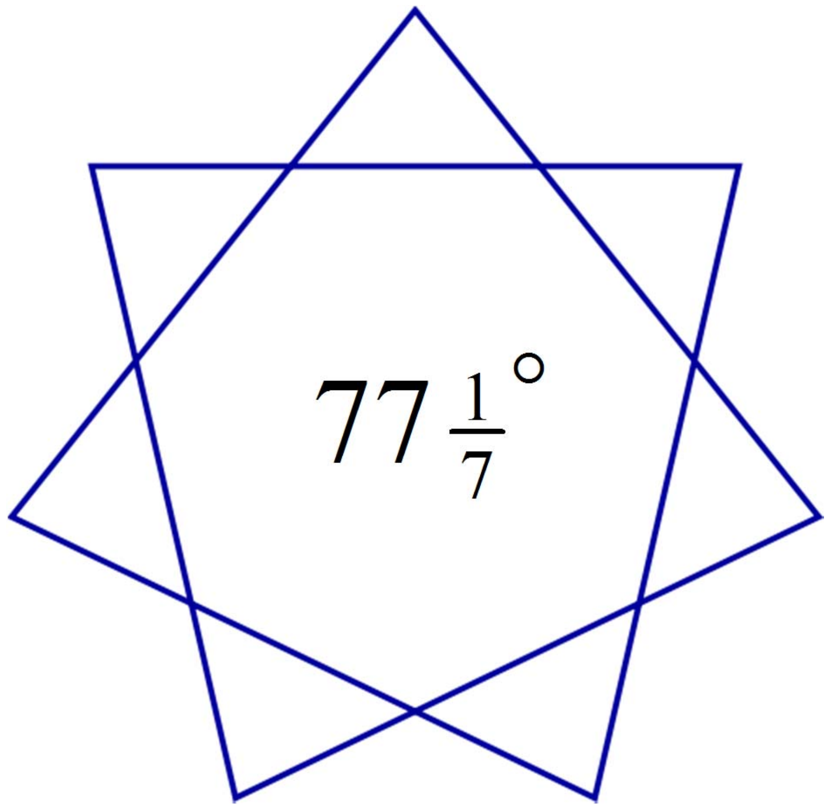
Familiar Five Pointed Star

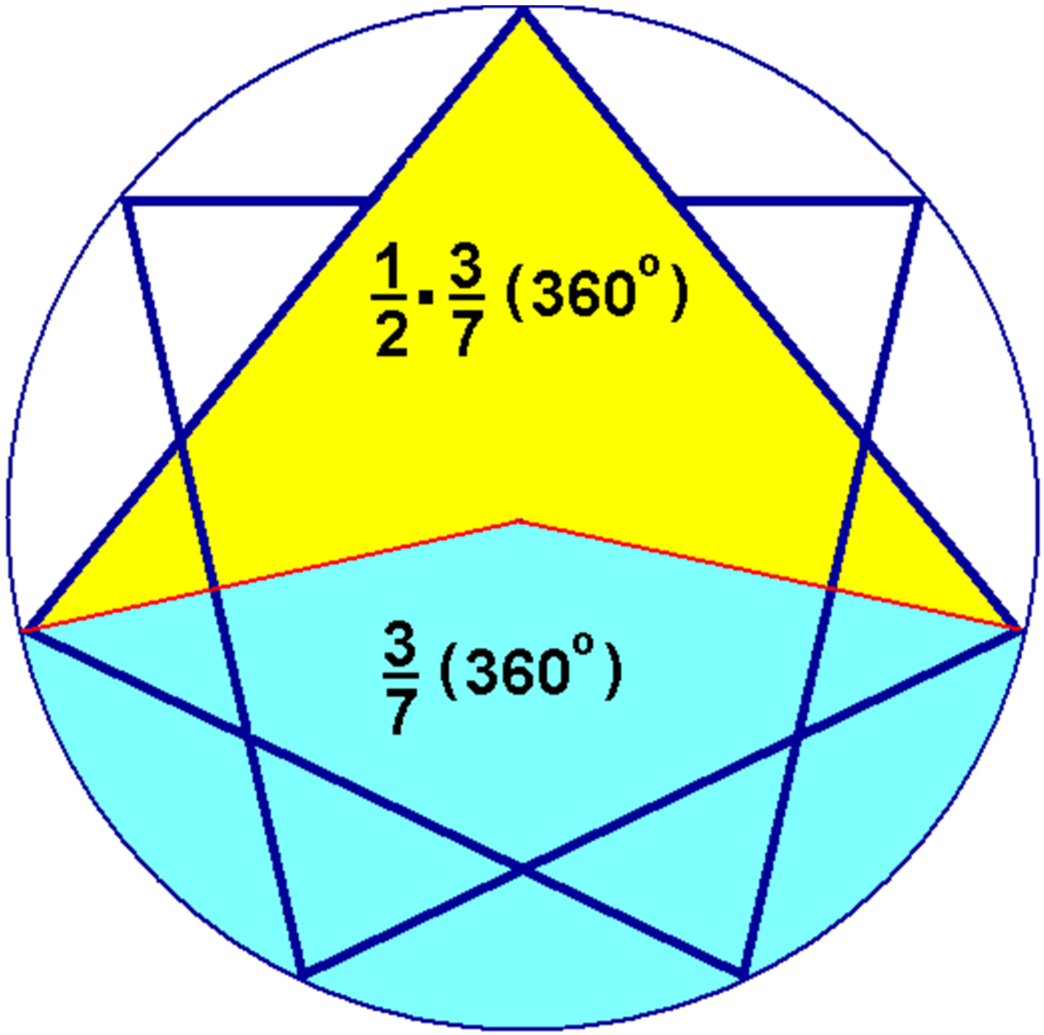


Q: What is the angle in each point?

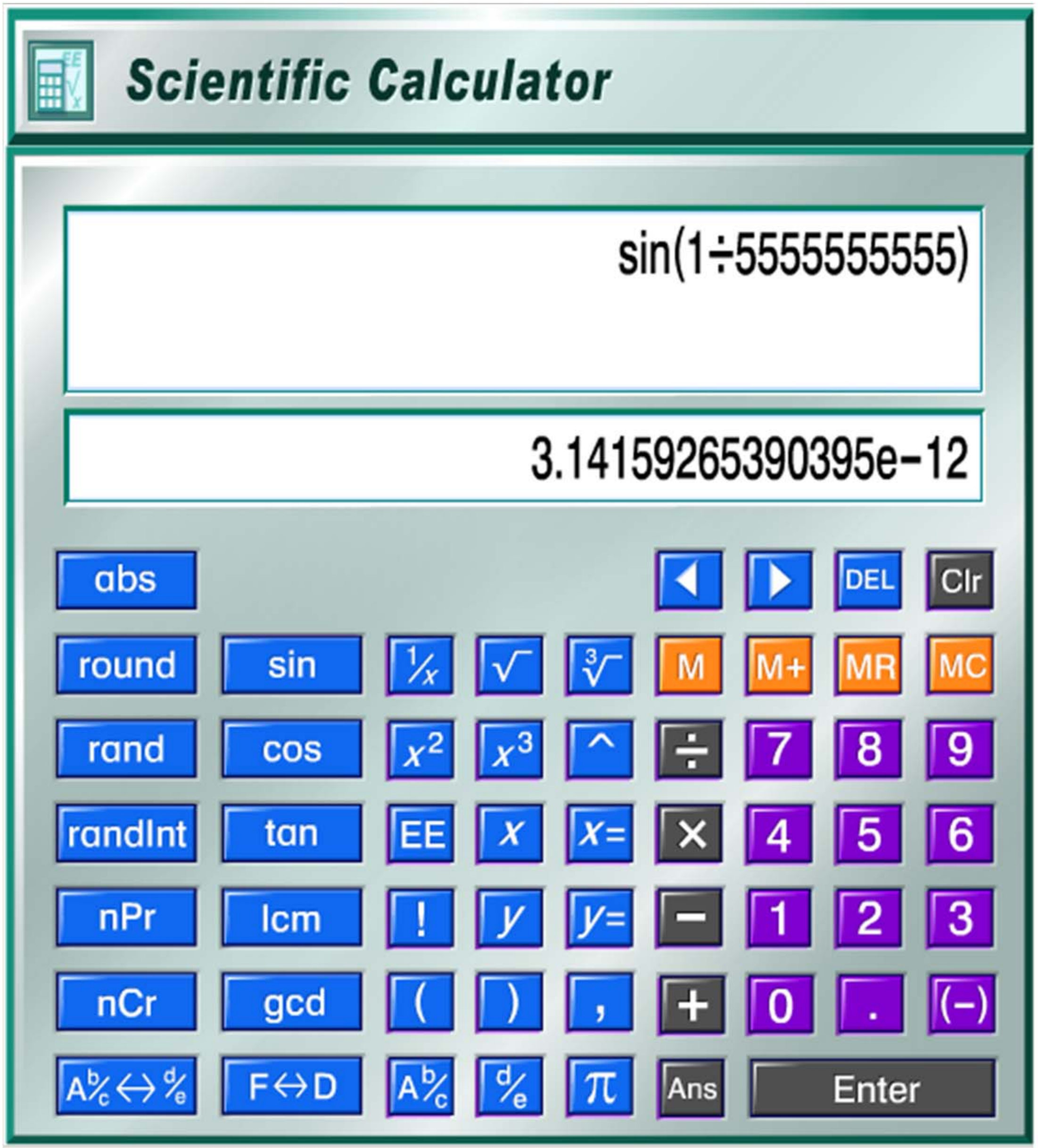
A: 36°

What's the angle for a 7 pointed star?




$\frac{1}{2} \cdot \frac{3}{7} (360^{o})$
$\frac{3}{7} (360^{o})$

sin 222222


Scientific Calculator
sin(1÷5555555555)
3.14159265390395e−12
abs
DEL
Clr
round
sin
M
M+
MR
MC
rand
cos
^
÷
7
8
9
randInt
tan
EE
×
4
5
6
nPr
lcm
!
−
1
2
3
nCr
gcd
(
)
,
+
0
.
(−)
F↔D
π
Ans
Enter

Explanation

- $.555\cdots = 5/9$
- $555\cdots 5 \approx 10^k\,(5/9)$
- $1/555\cdots 5 \approx 10^{-k}\,(9/5)$
- $(\pi/180)(1/555\cdots 5) \approx (\pi/180)\,(10^{-k})(9/5)$

$$\approx \pi(10^{-k})(9/900)$$

$$\approx \pi(10^{-k-2})$$

- $\sin((\pi/180)(1/555\cdots 5)) \approx \sin(\pi\cdot 10^{-k-2})$

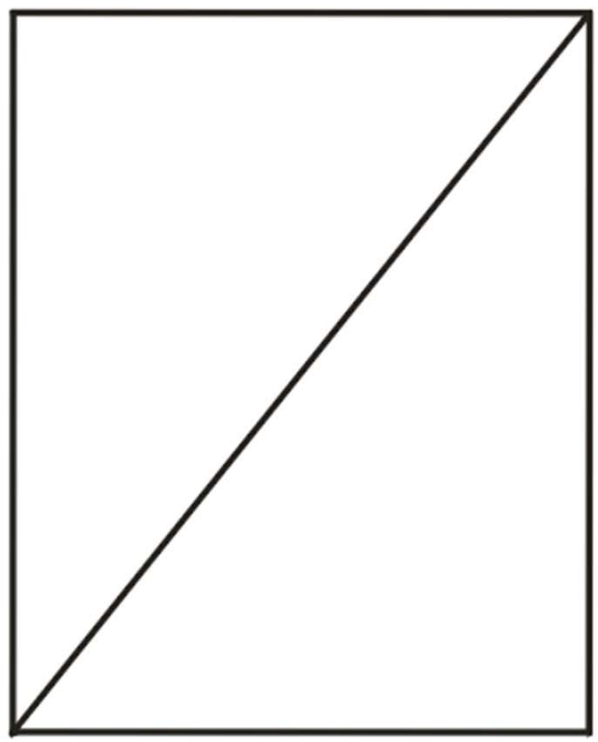
$$\approx \pi\cdot 10^{-k-2}$$

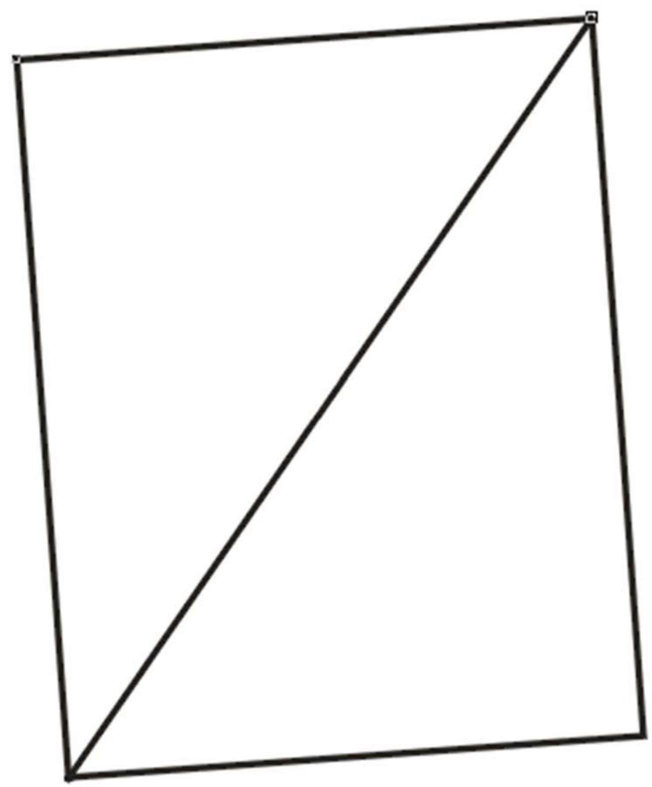
Sin(10^k)

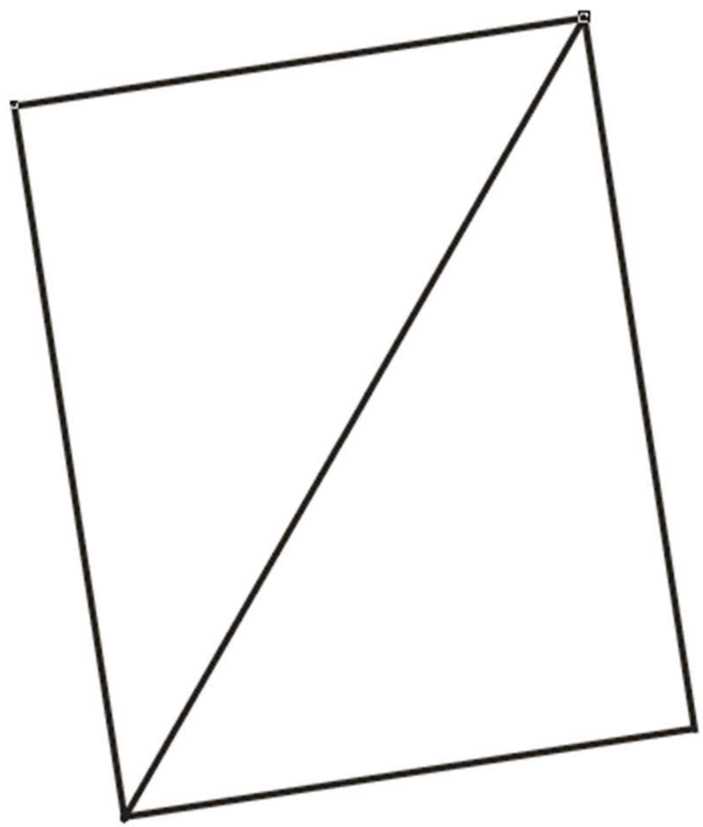
- If the angle is in degrees, the sequence converges. Find the limit.
- If the angle is in radians, does the sequence converge?

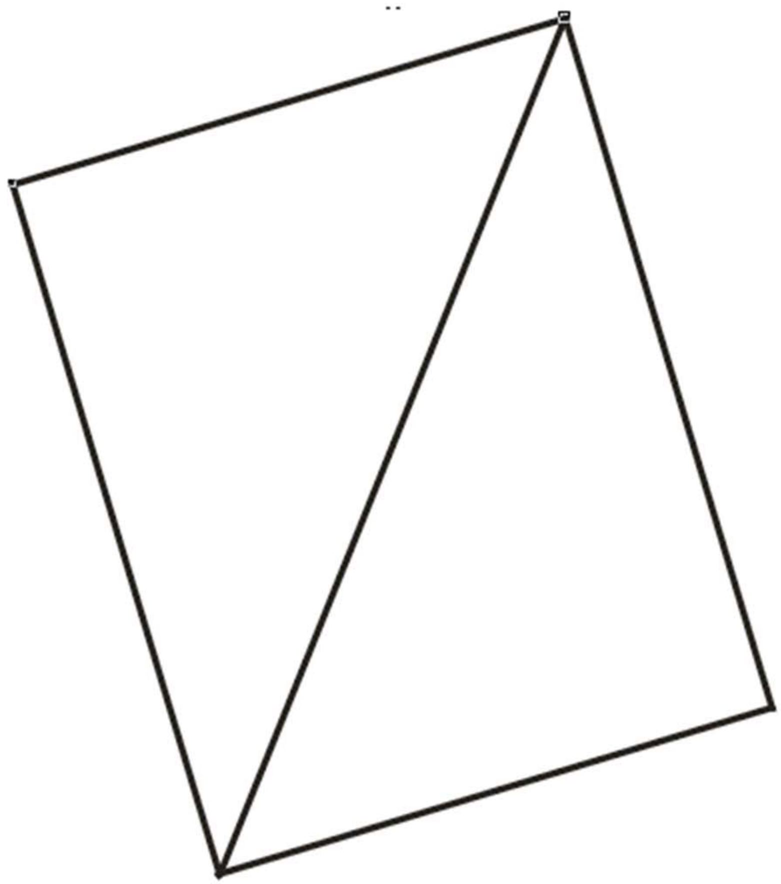
(I don't think so -- Equidistribution theory)

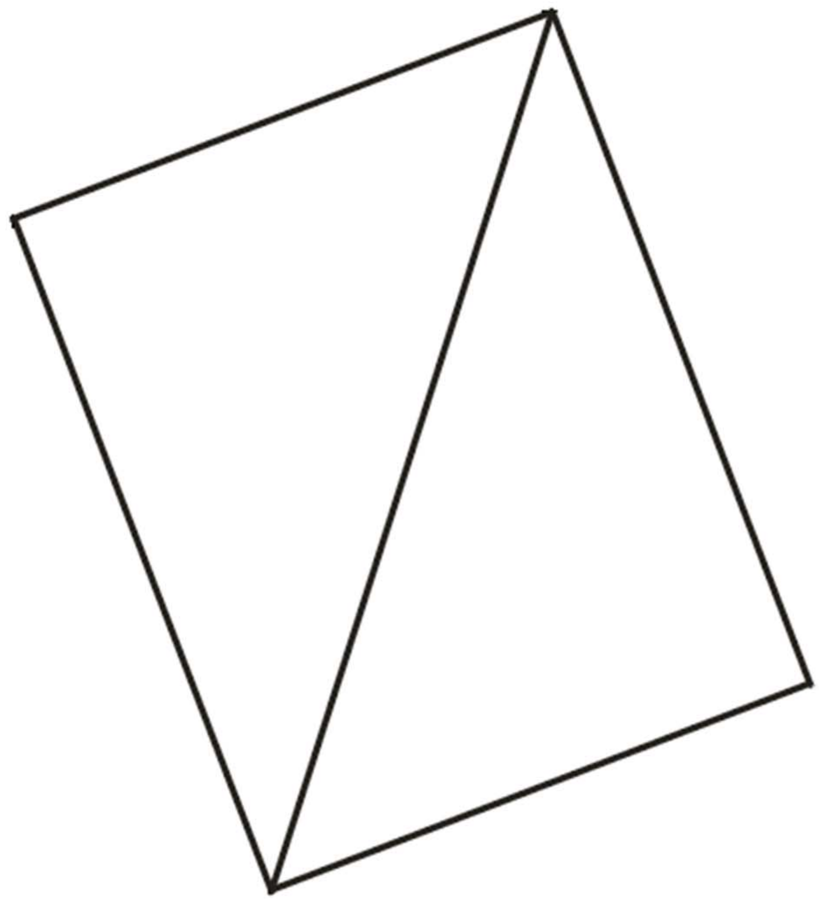
Sum Formulas for Sine and Cosine

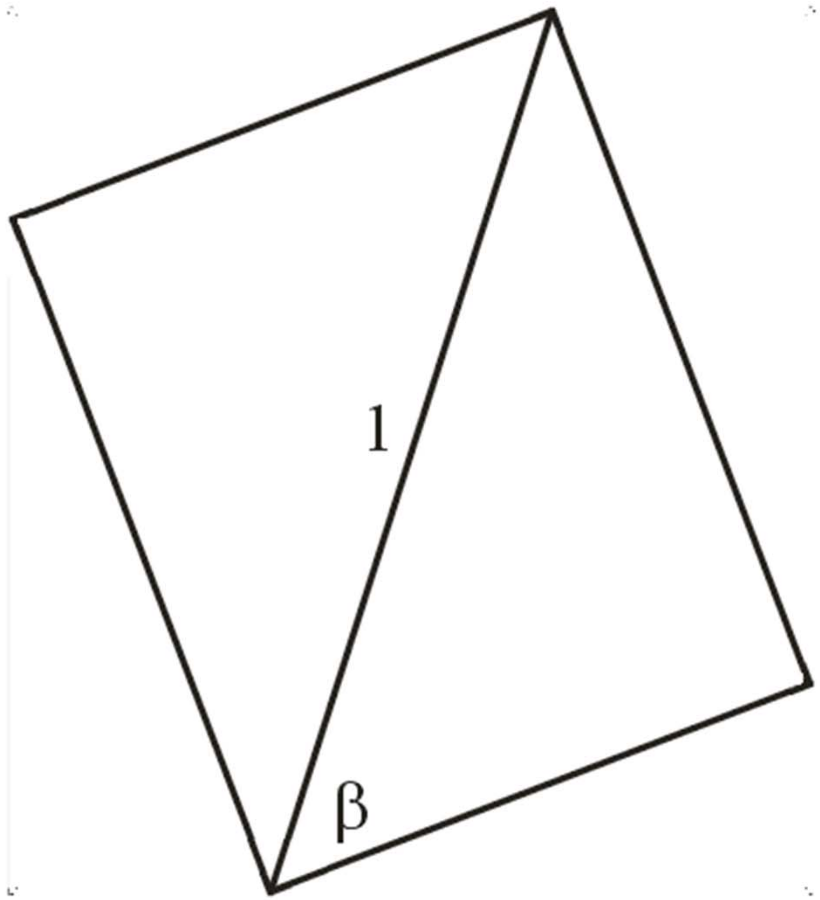


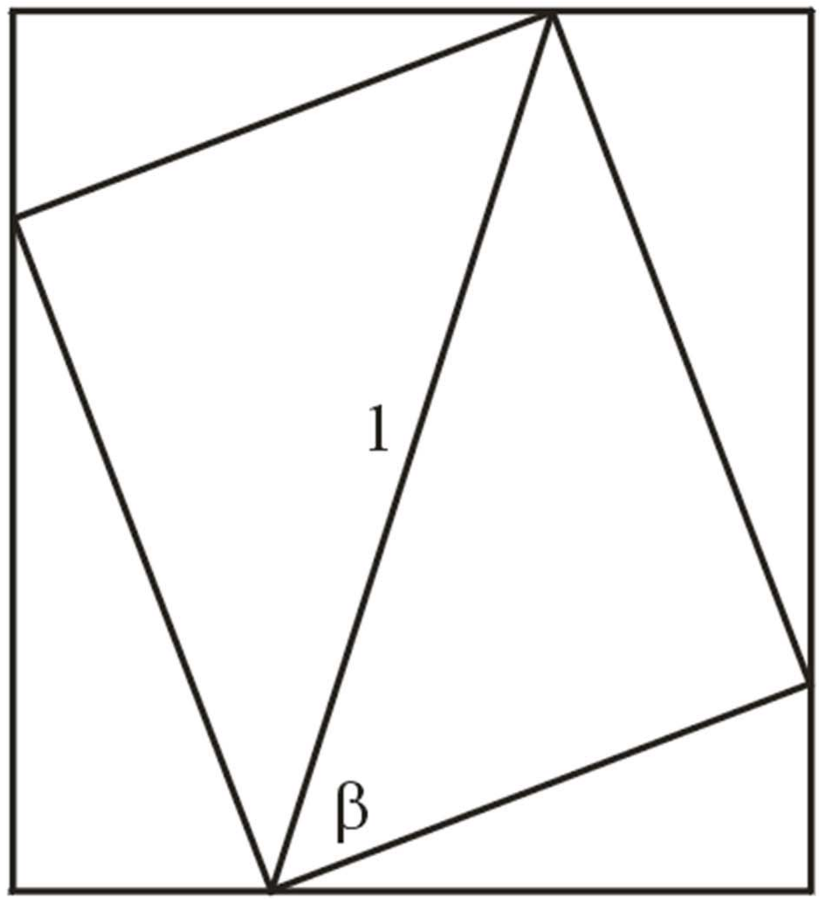


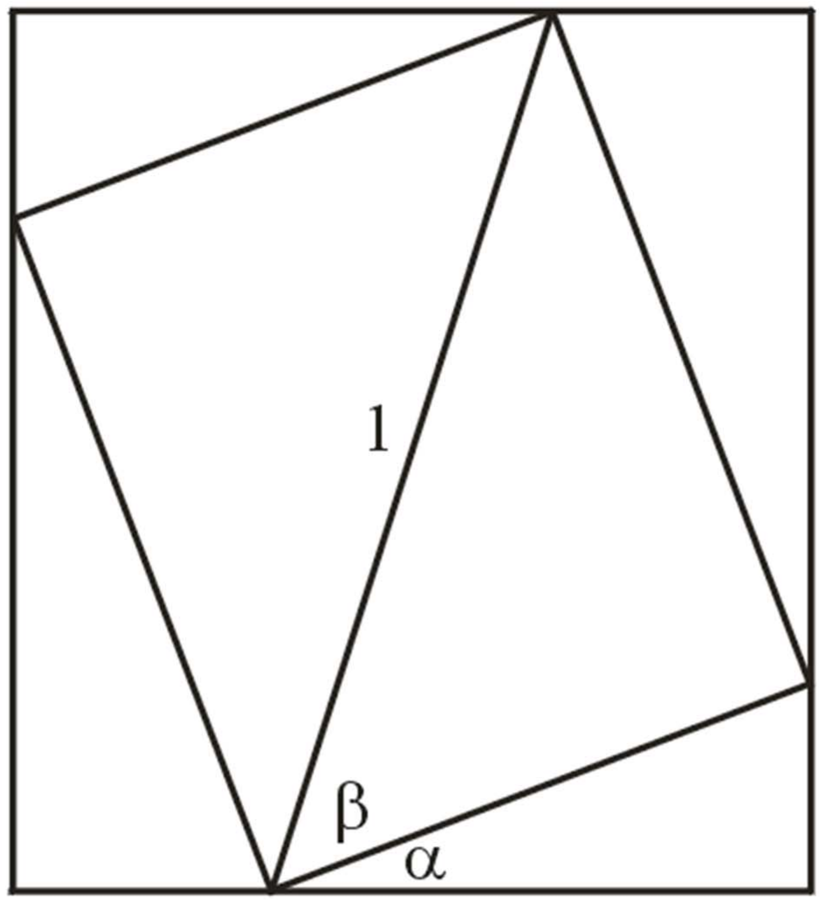


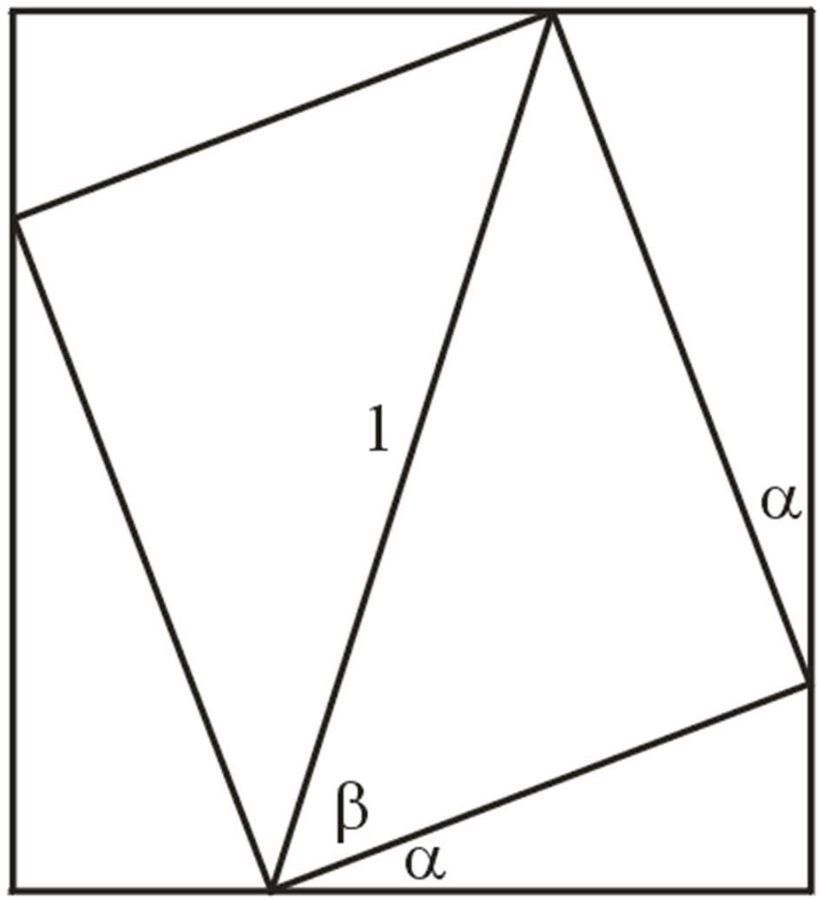


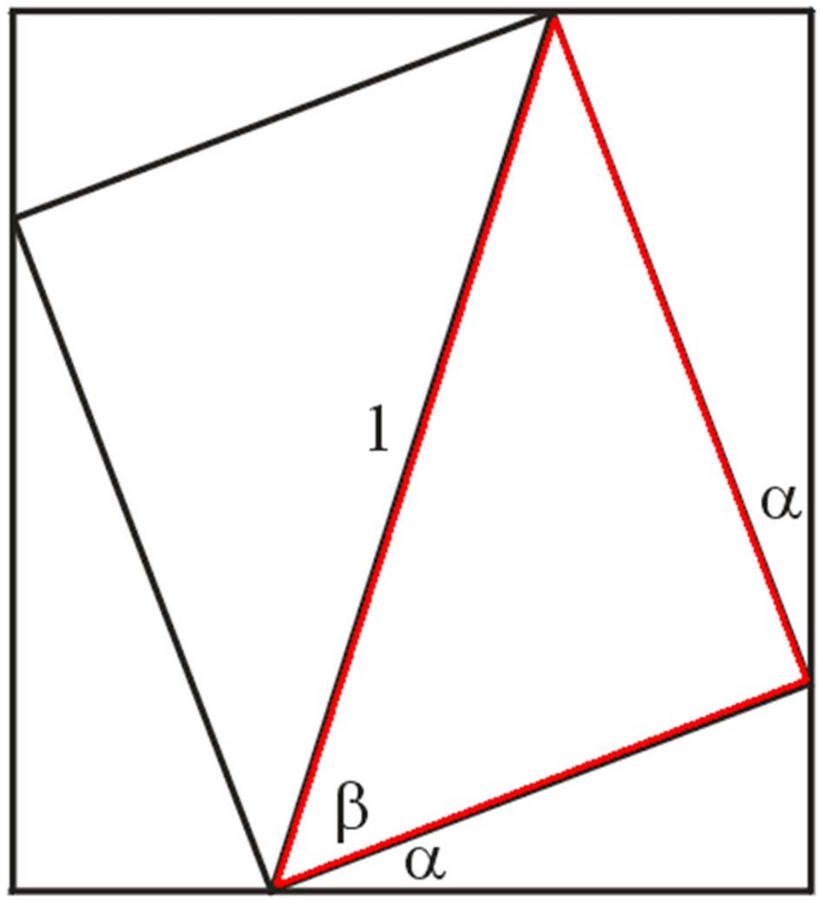


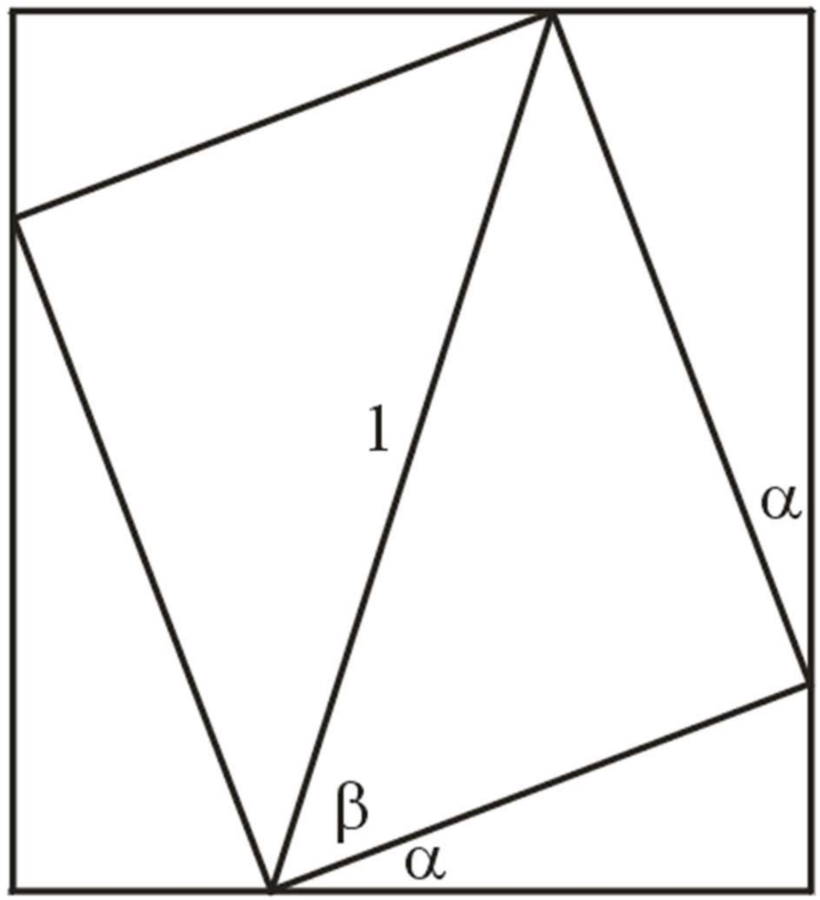

1
β

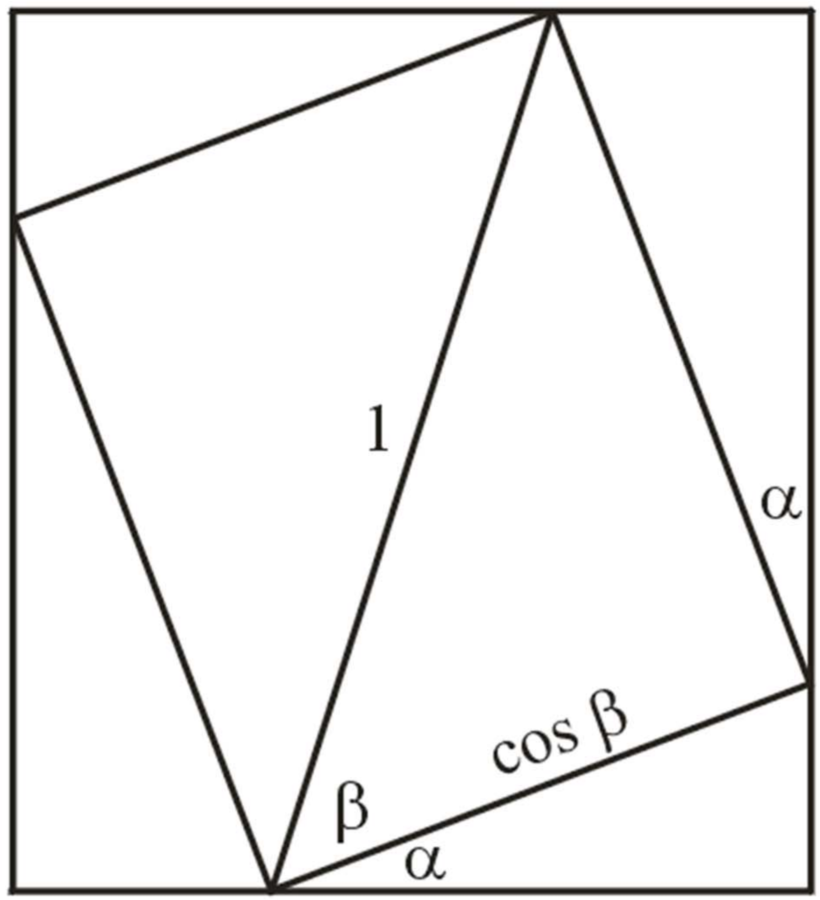

1
β

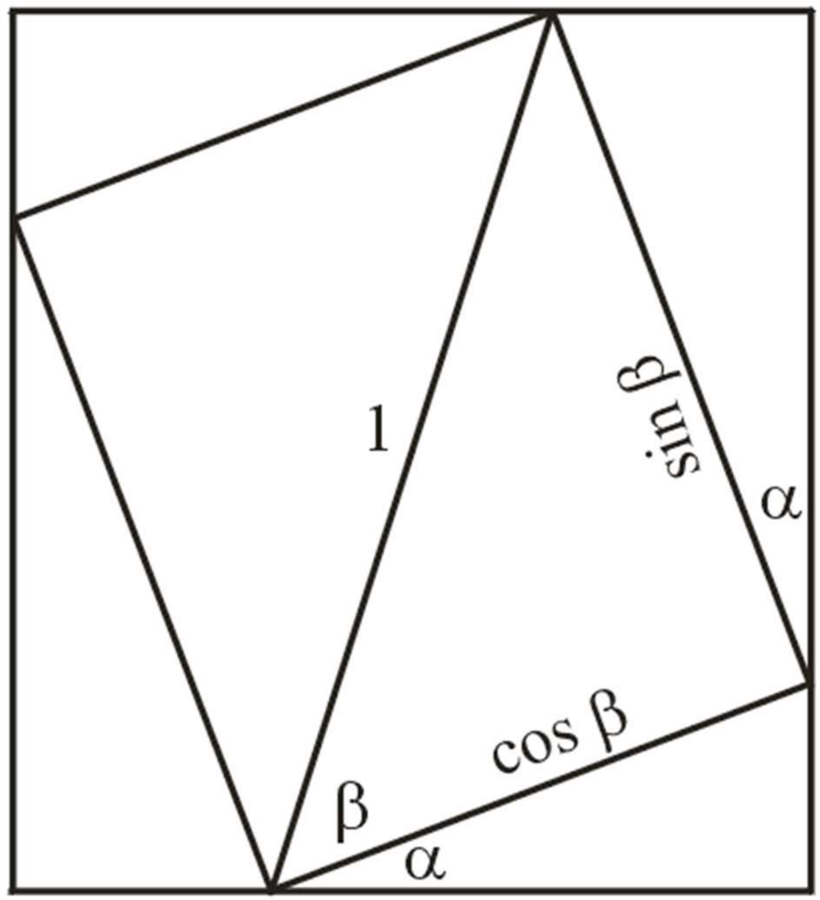

1
β
α


1
α
β
α

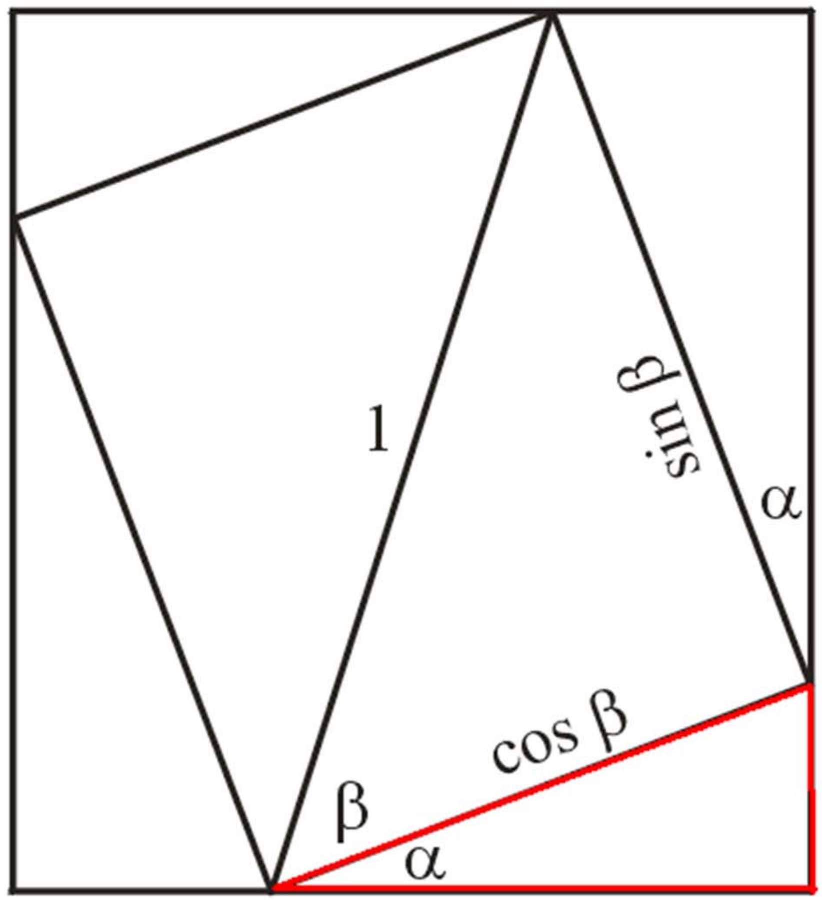

1
α
β
α


1
α
β
α

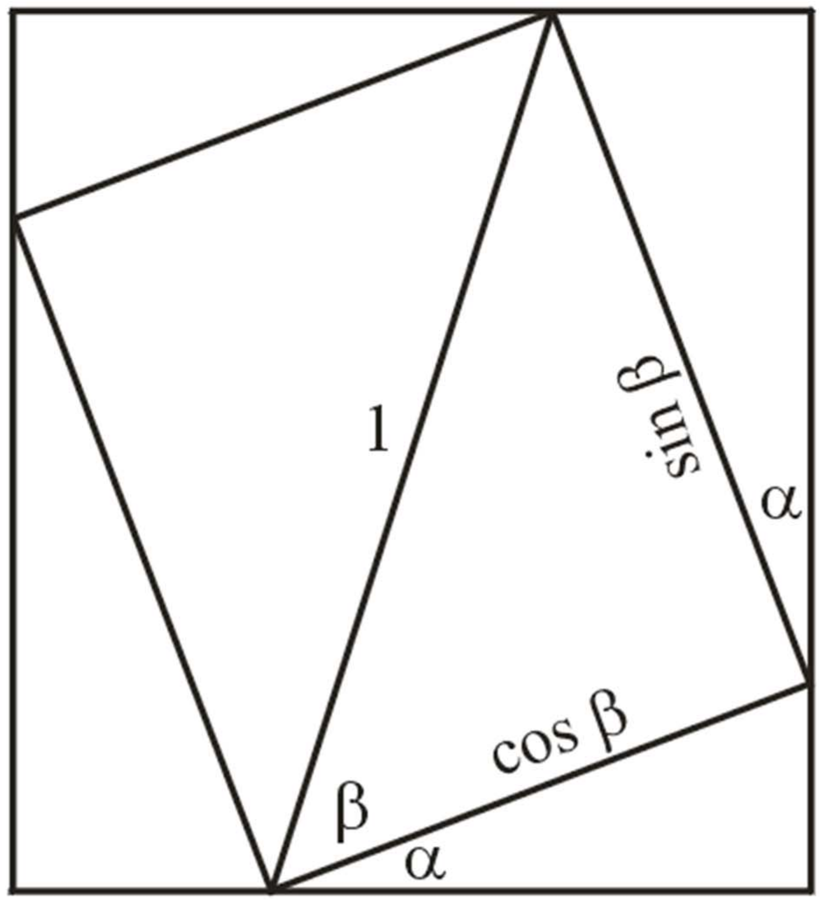

1
α
cos β
β
α

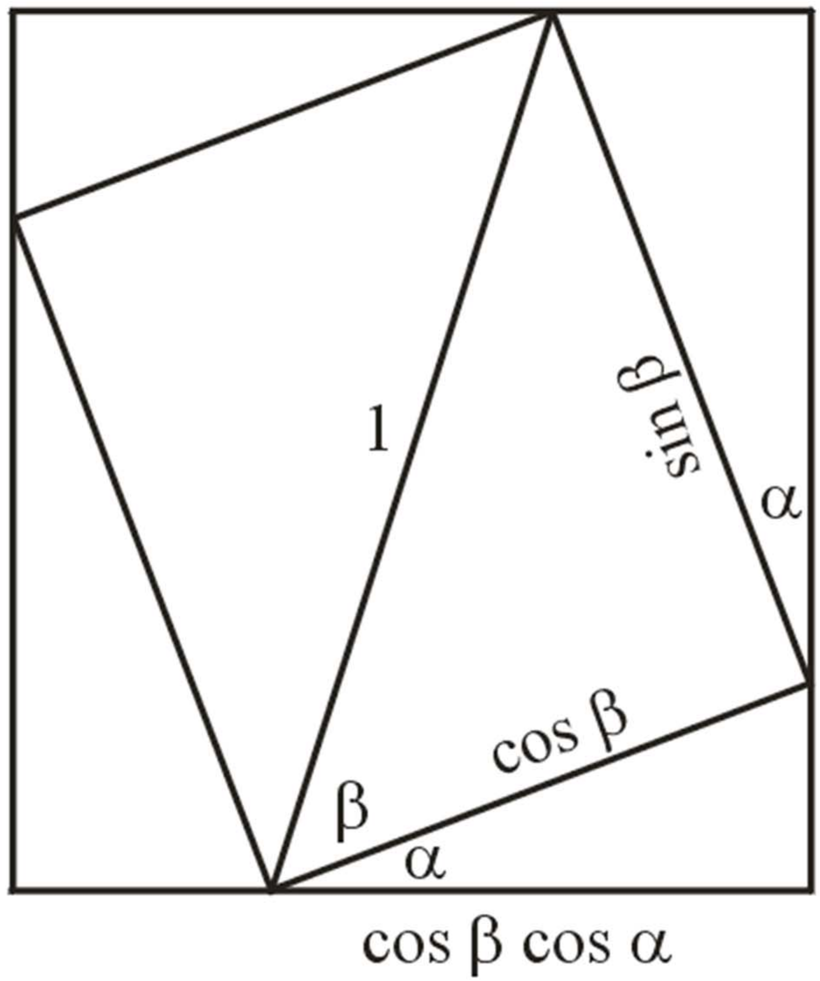


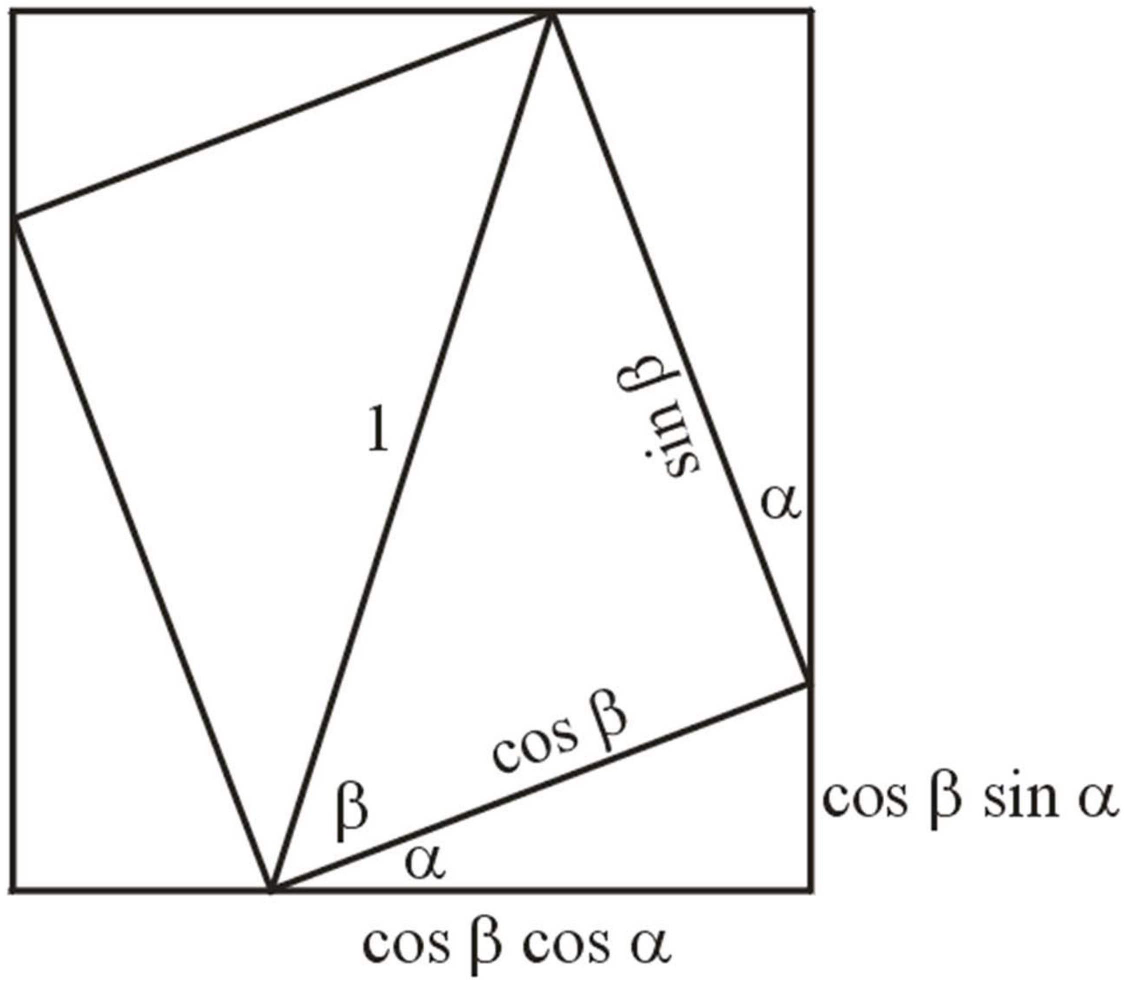
1
sin β
α
cos β
β
α

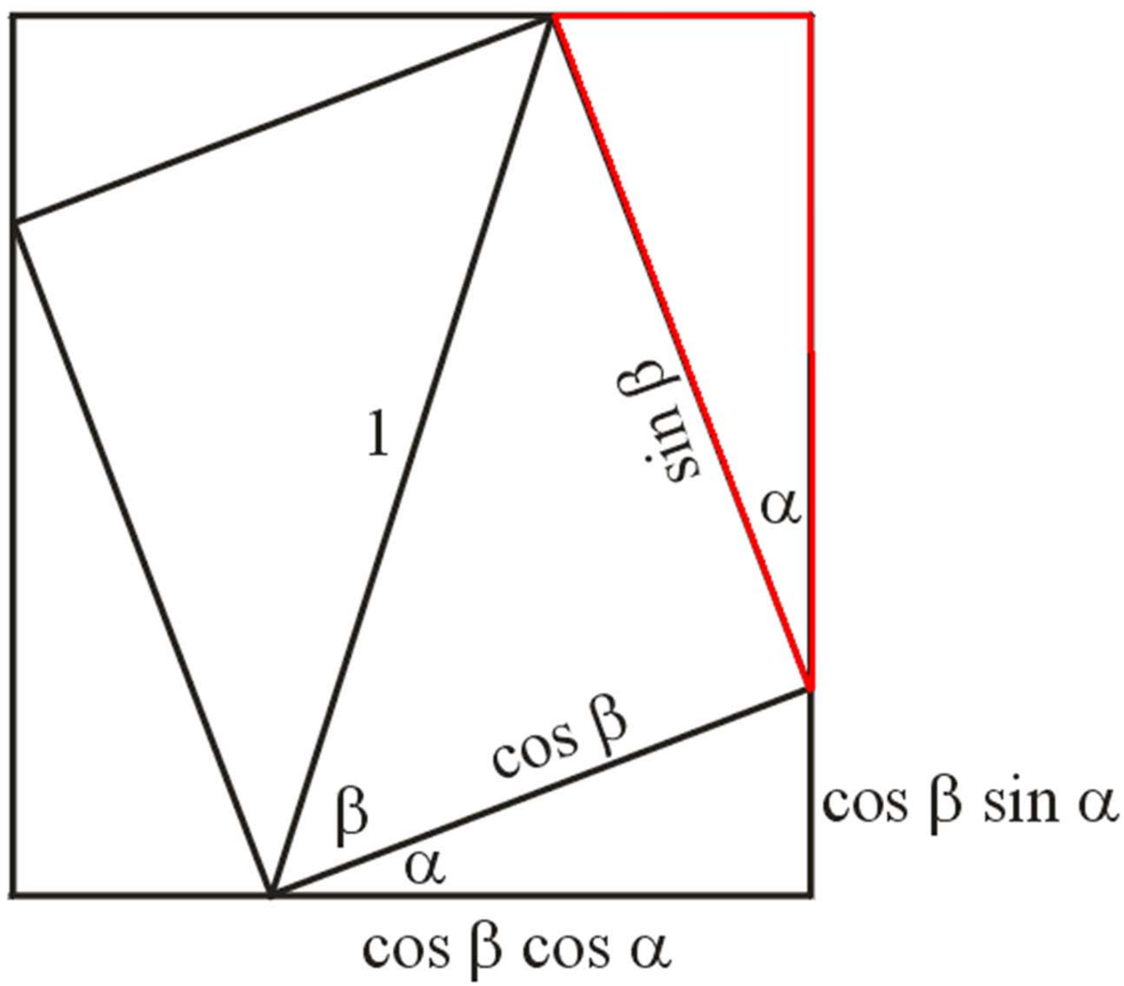


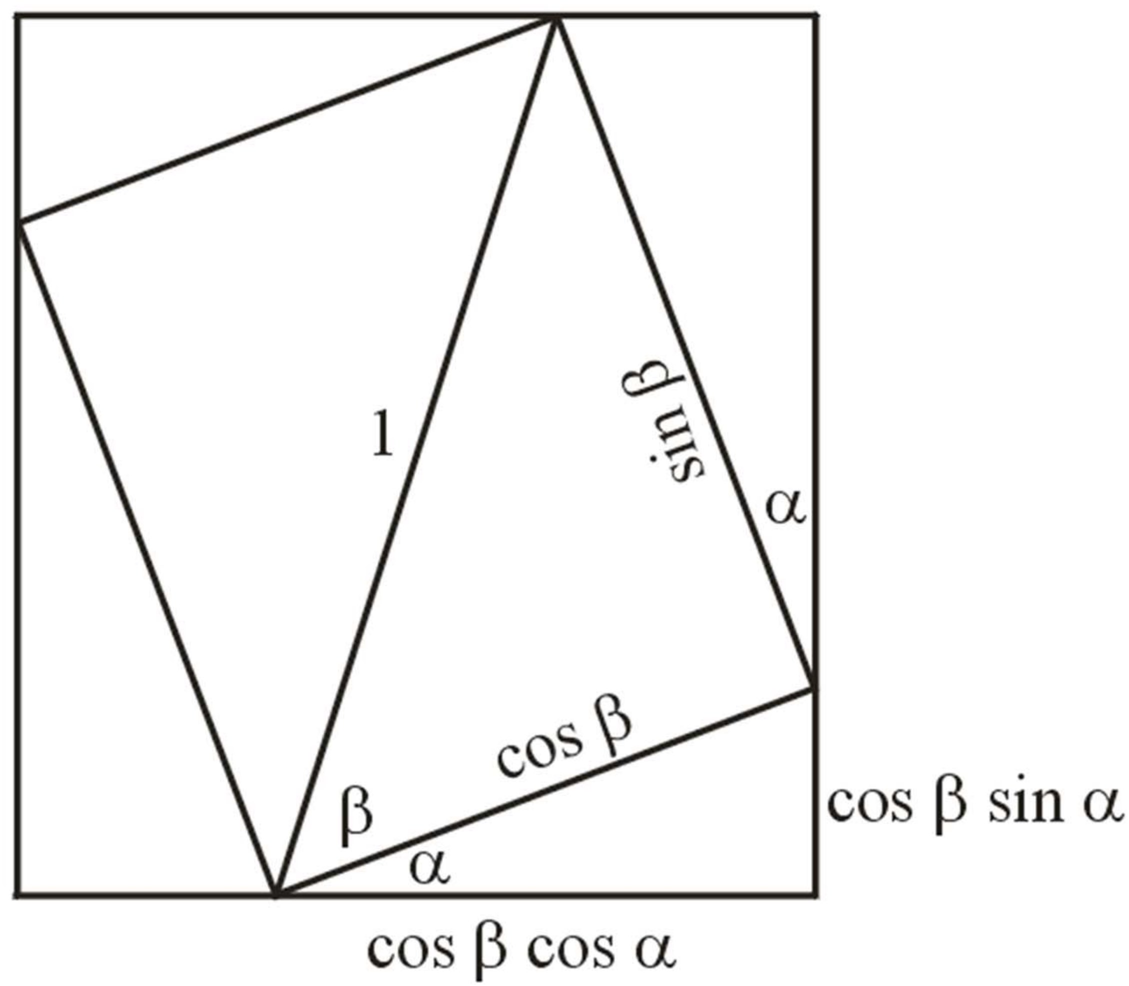
1
sin β
α
cos β
β
α


1
sin β
α
cos β
β
α

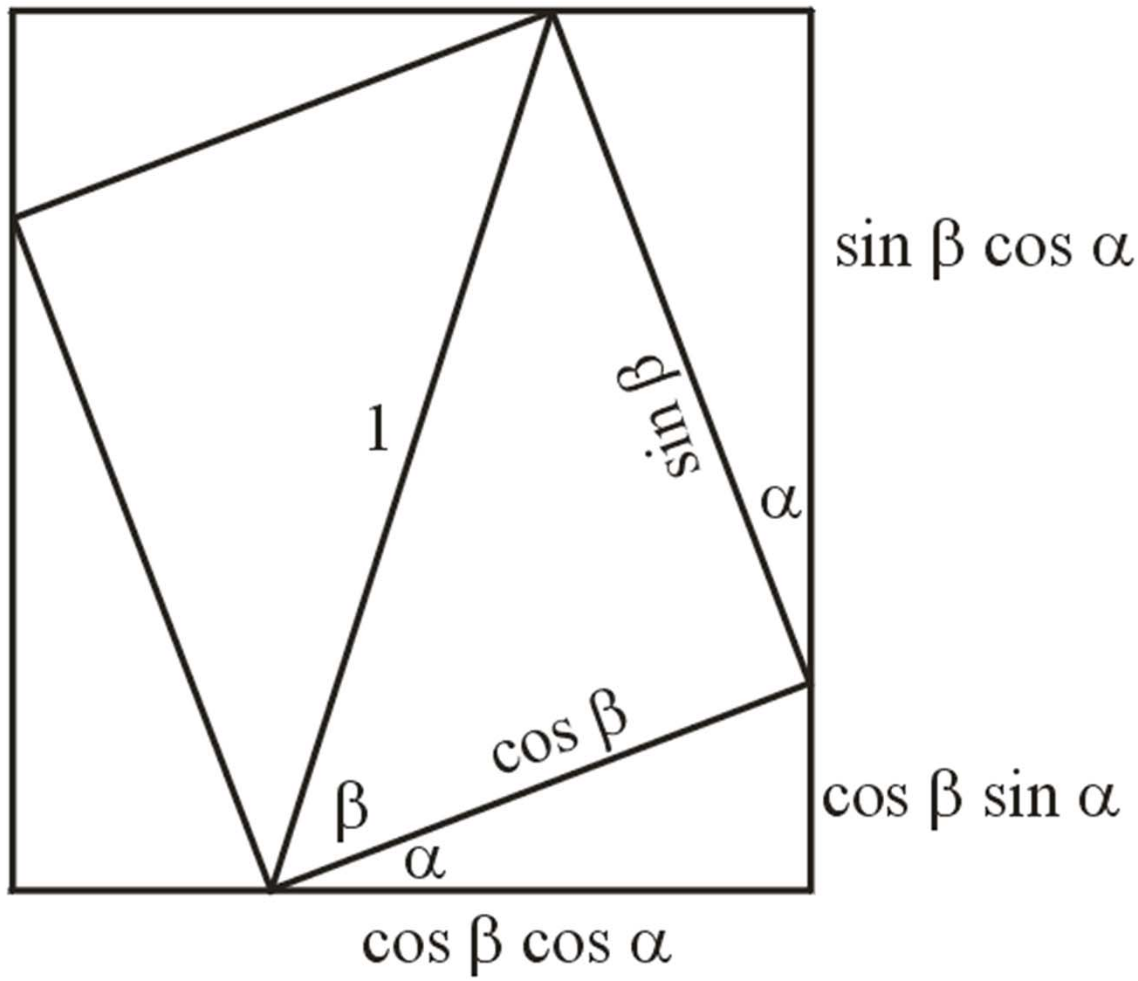

1
sin β
α
cos β
β
α
cos β cos α


1
sin β
α
cos β
β
α
cos β sin α
cos β cos α

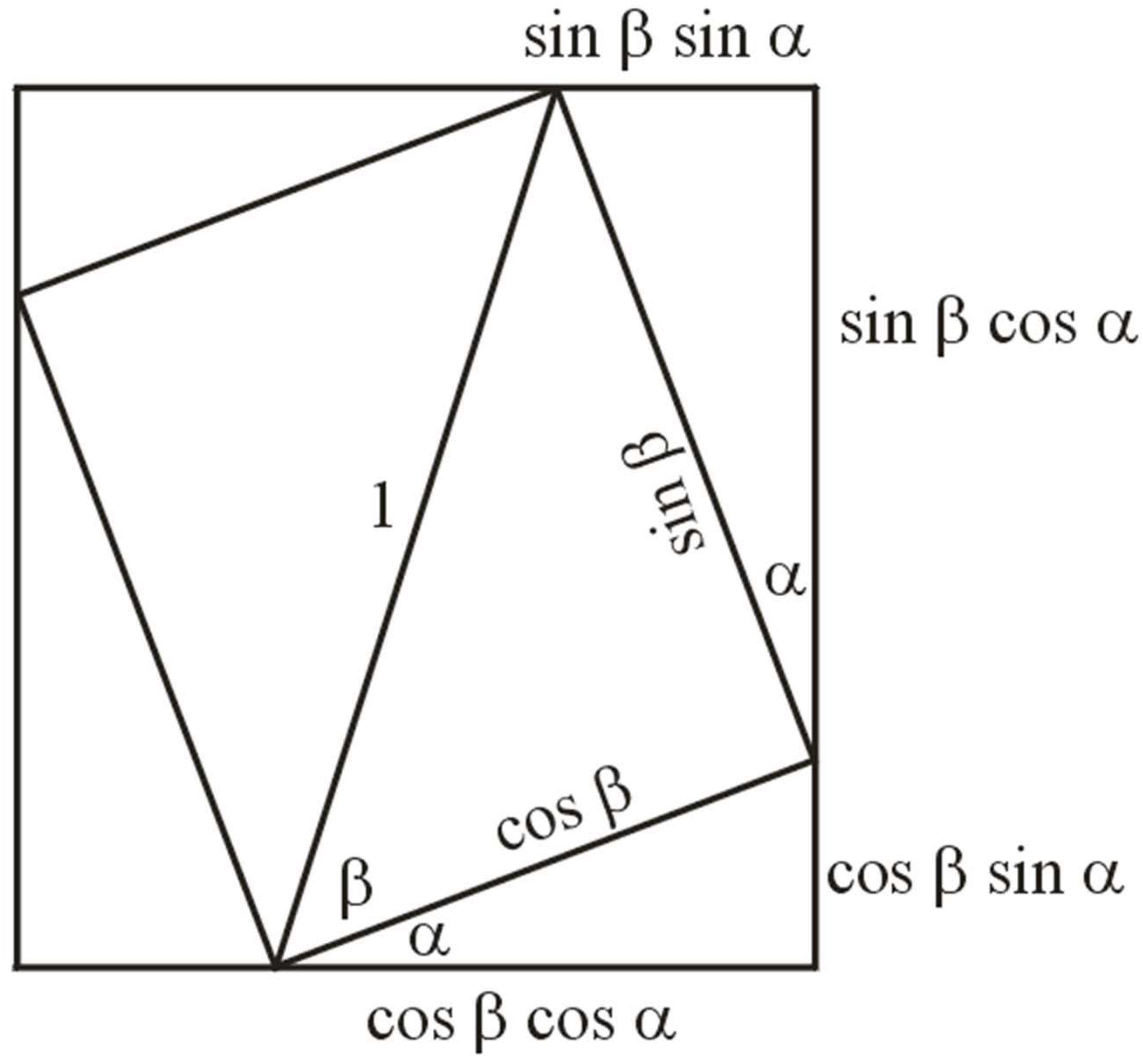

1
sin β
α
cos β
β
α
cos β sin α
cos β cos α

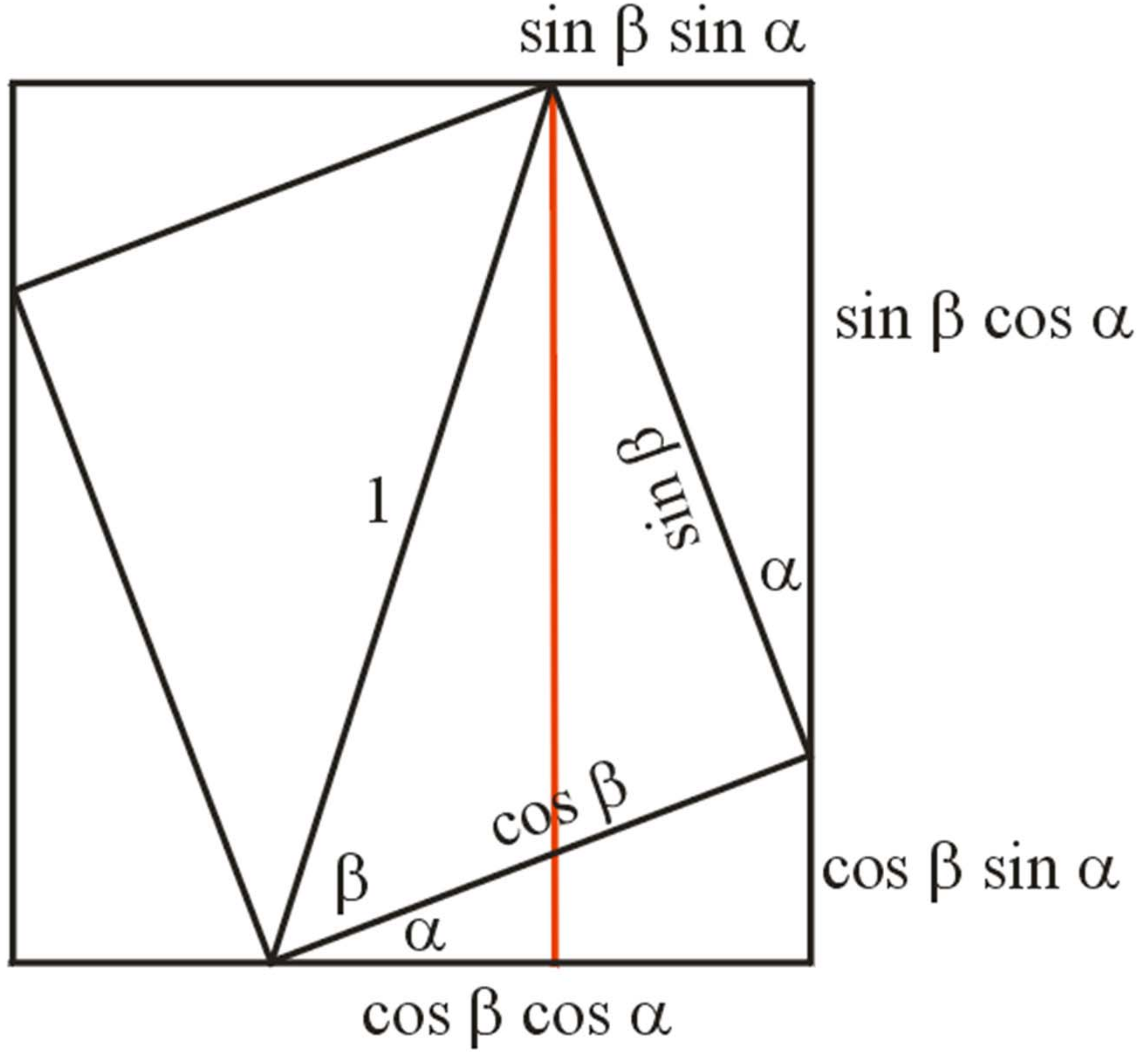


1
sin β
α
cos β
β
α
cos β sin α
cos β cos α

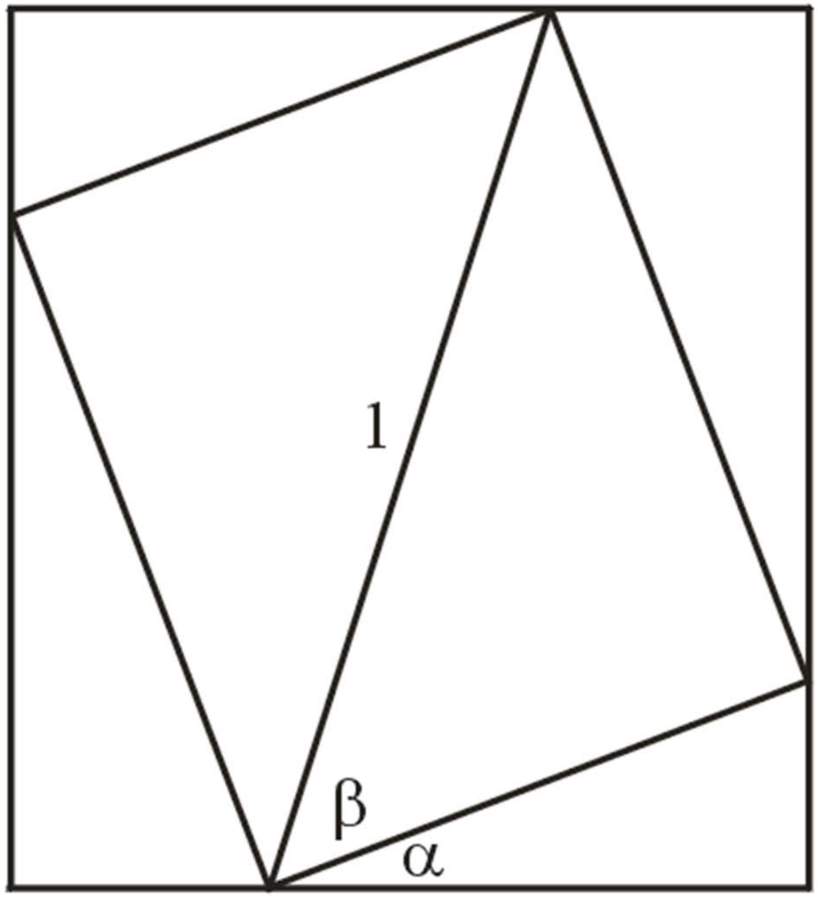


sin β cos α
1
sin β
α
cos β
β
α
cos β sin α
cos β cos α


sin β sin α
sin β cos α
1
sin β
α
cos β
β
α
cos β sin α
cos β cos α



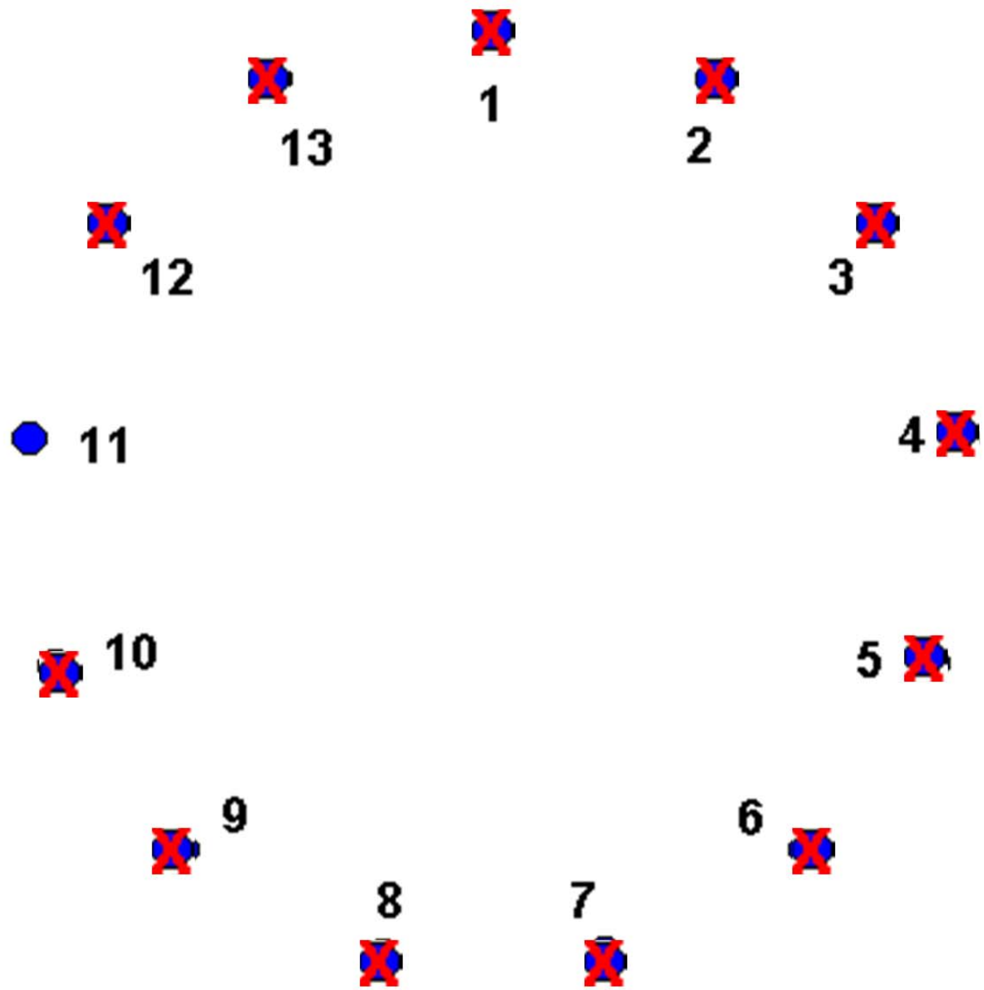
sin β sin α
sin β cos α
1
sin β
α
cos β
β
α
cos β sin α
cos β cos α



BEHOLD!

Josephus Problem

Graham, Knuth, Patashnik, *Concrete Mathematics*


1
2
3
4
5
6
7
8
9
10
11
12
13

Where should you stand?

- Suppose there are *n* people in the circle
- Which position will be the last remaining?
- Solution: Write *n* in binary

$$13 \rightarrow 1101$$

- Shift the left-most digit to the right end

$$1101 \rightarrow 1011$$

- Convert back to base 10

$$1011 \rightarrow 11$$

Parity
of
Pascal

How Many Odd Numbers in Row *n* of Pascal's Triangle?

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

How Many Odd Numbers in Row *n* of Pascal's Triangle?

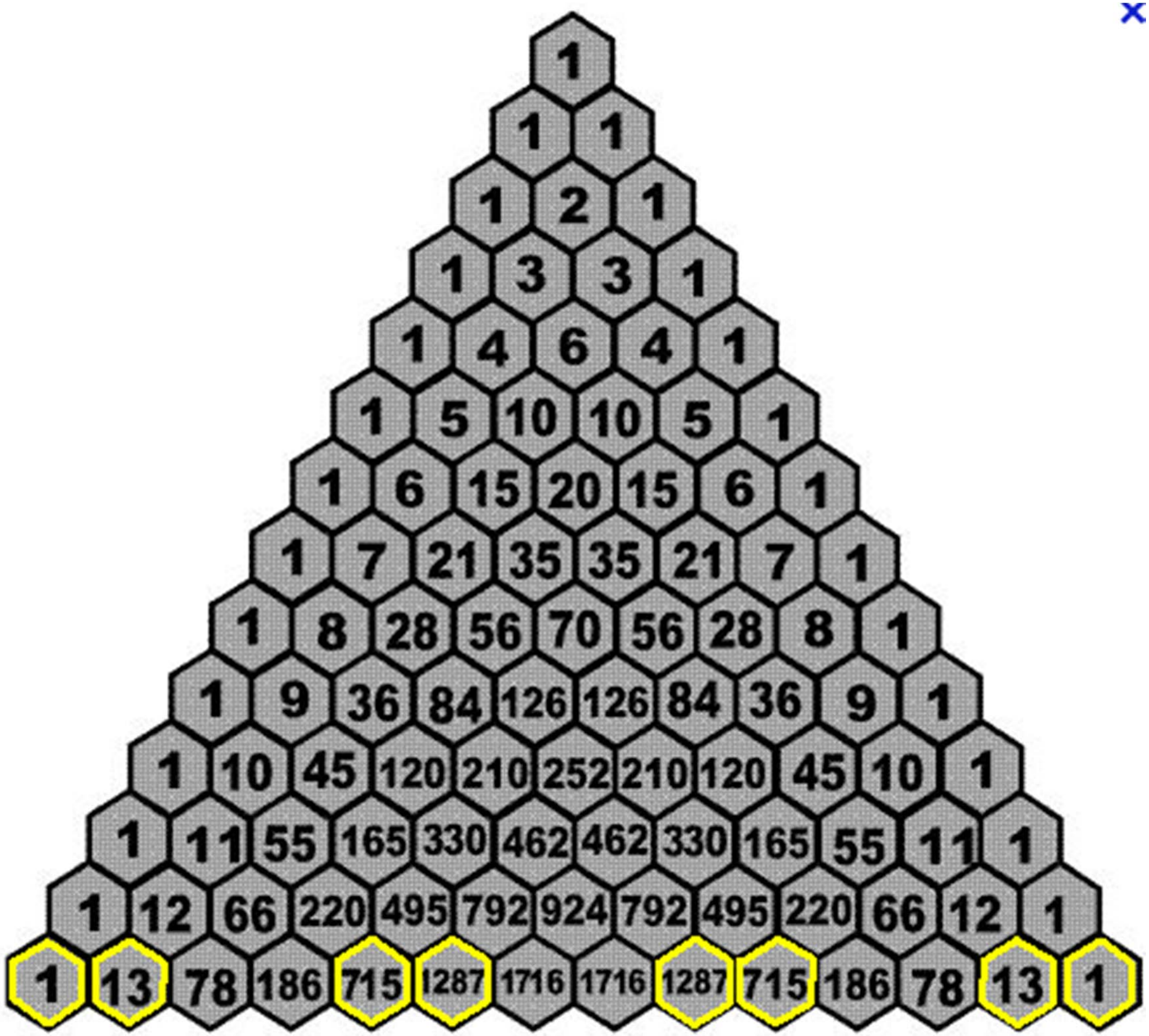
	row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5

How Many Odd Numbers in Row *n* of Pascal's Triangle?

Pascal's Triangle	row	odds
1	0	1
1 1	1	2
1 2 1	2	2
1 3 3 1	3	4
1 4 6 4 1	4	2
1 5 10 10 5 1	5	4

Solution

- Express *n* in binary
- Count the 1's to find *m*
- The number of odds is 2^m
- Example: $n = 13$
- Binary form for 13 is 1101 so $m = 3$
- $2^3 = 8$ odd numbers in the 13^{th} row of Pascal's triangle


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 186 715 1287 1716 1716 1287 715 186 78 13 1

What happens mod k?

What happens mod *k*?

- I stumbled on this problem ca. 1980
- Library research: papers in maa pubs about every 25 years for past 100 years or more.
- I was right on time!
- Another cycle in the early 2000's launched ***The PascGalois Project: Visualizing Abstract Mathematics*** right here at SU.
- Generalized problem, NSF support, student involvement, several of our MAA friends: Bardzell, Shannon, Spickler, Bergner, *et al*.

- http://faculty.salisbury.edu/~despickler/pascgalois/

Fraction Addition Made Difficult

To add 2/3 and 5/8 …

For differentiable f and g,

$$\frac{f'(x)}{f(x)} = (\ln f(x))' \text{ and } \frac{g'(x)}{g(x)} = (\ln g(x))'$$

$$\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} = (\ln f(x)g(x))' \quad = \frac{[f(x)g(x)]'}{f(x)g(x)}$$

$$= \frac{f'(x)g(x) + f(x)g(x)'}{f(x)g(x)}$$

$$\frac{f'(0)}{f(0)} + \frac{g'(0)}{g(0)} = \frac{f'(0)g(0) + f(0)g'(0)}{f(0)g(0)}$$

Define $f(x) = 2x + 3$ and $g(x) = 5x + 8$

$$\frac{2}{3} + \frac{5}{8} = \frac{3 \cdot 5 + 2 \cdot 8}{3 \cdot 8}$$

Interlude: Haunted by Pythagoras

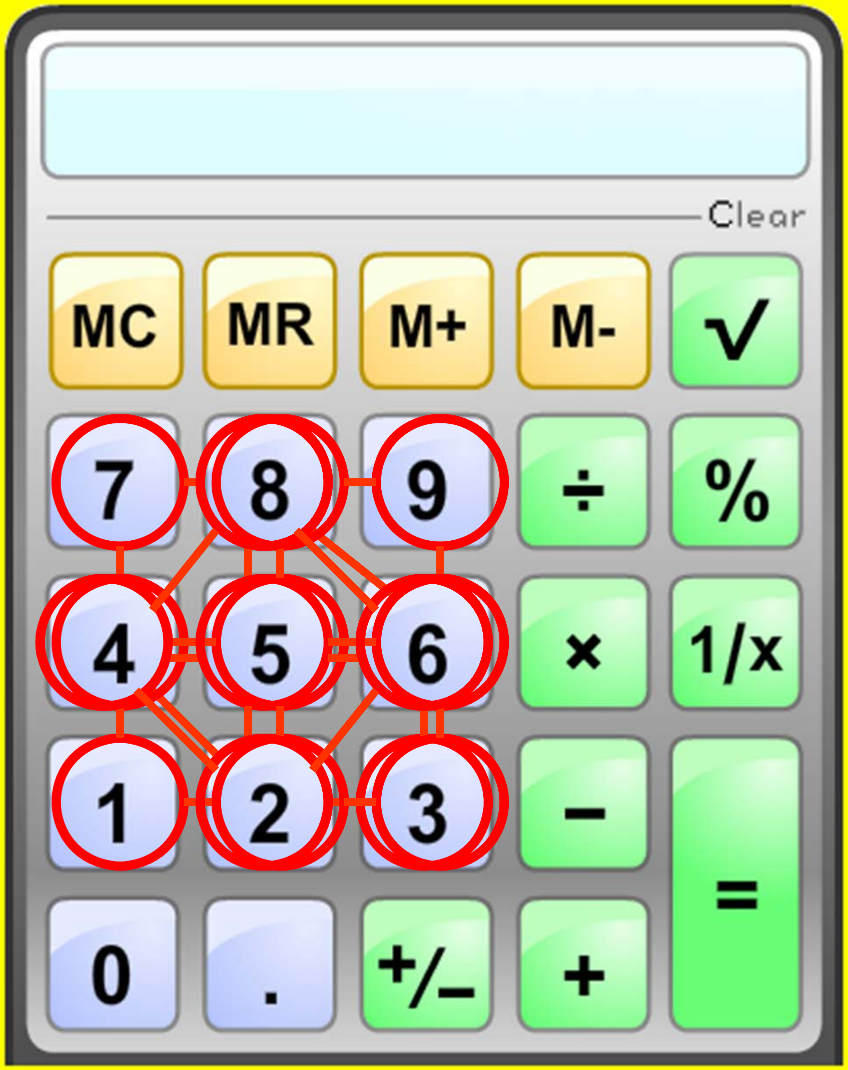
The Ghost of Pythagoras

My Favorite Coffee Drink

Pythagorean Cup

When I got back to my office…

Magic Circles


Clear
MC
MR
M+
M-
√
7
8
9
÷
%
4
5
6
×
1/x
1
2
3
−
=
0
.
+/_
+

A Cute Lill Theorem

Lill's Method

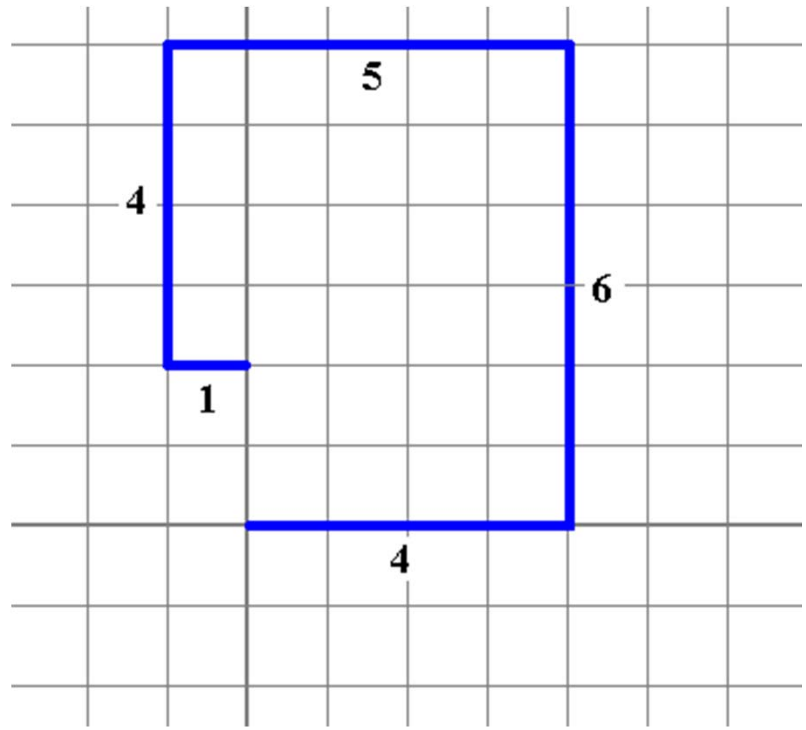
- Misnomer – not really a method for finding roots
- Geometric visualization of a root
- Lill was an Austrian military engineer
- Published his method in 1867
- More recently this method has received renewed interest in connection with origami

Lill's Method Example

- Goal: find a root of

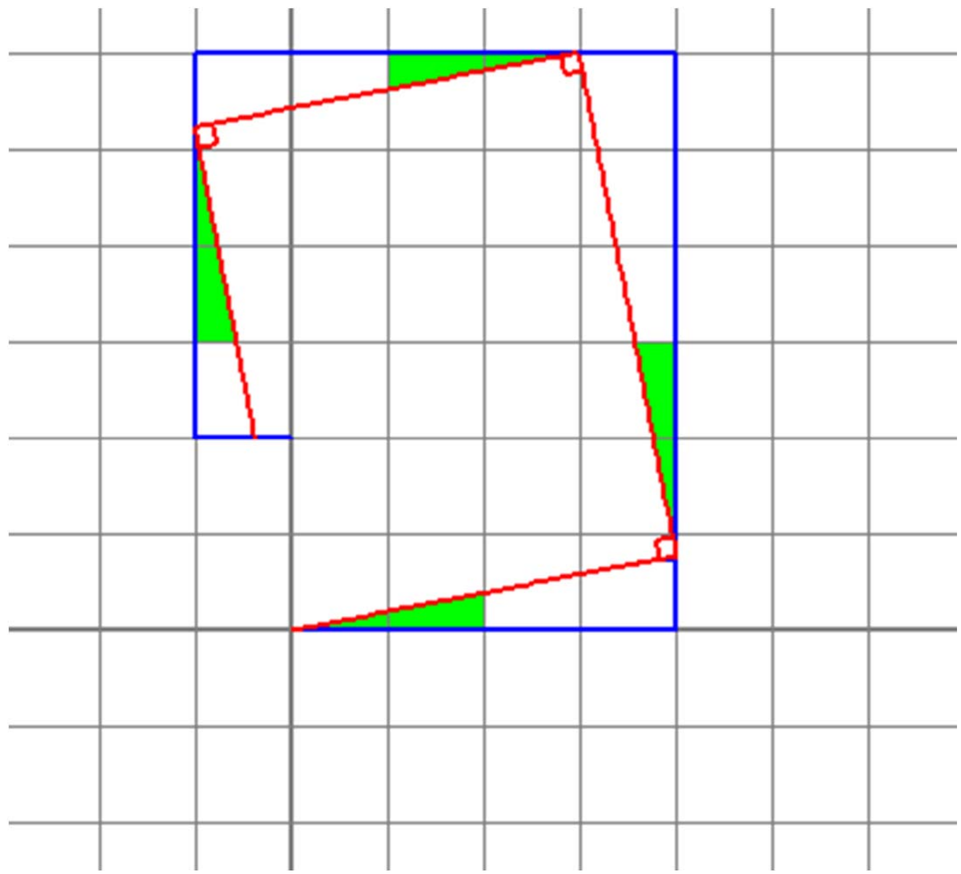
$$p(t) = 4t^4 + 6t^3 + 5t^2 + 4t + 1$$

- Use coefficients to construct a right polygonal path (the Primary Lill Path).



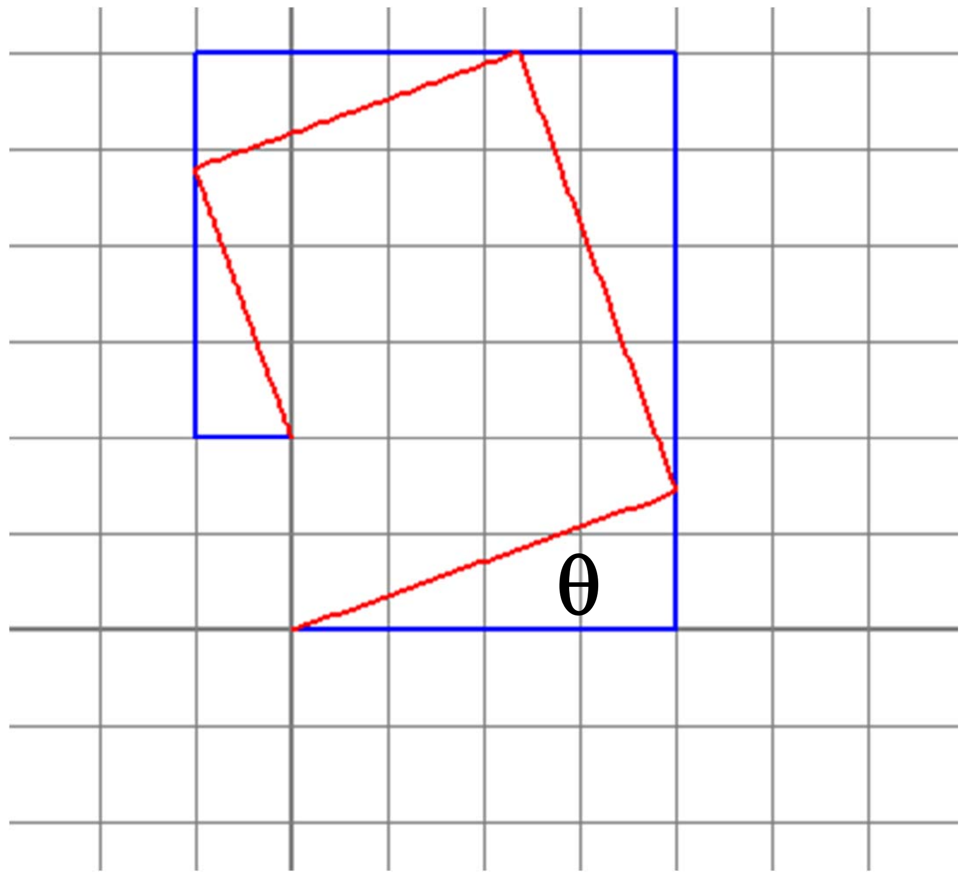
Secondary Path

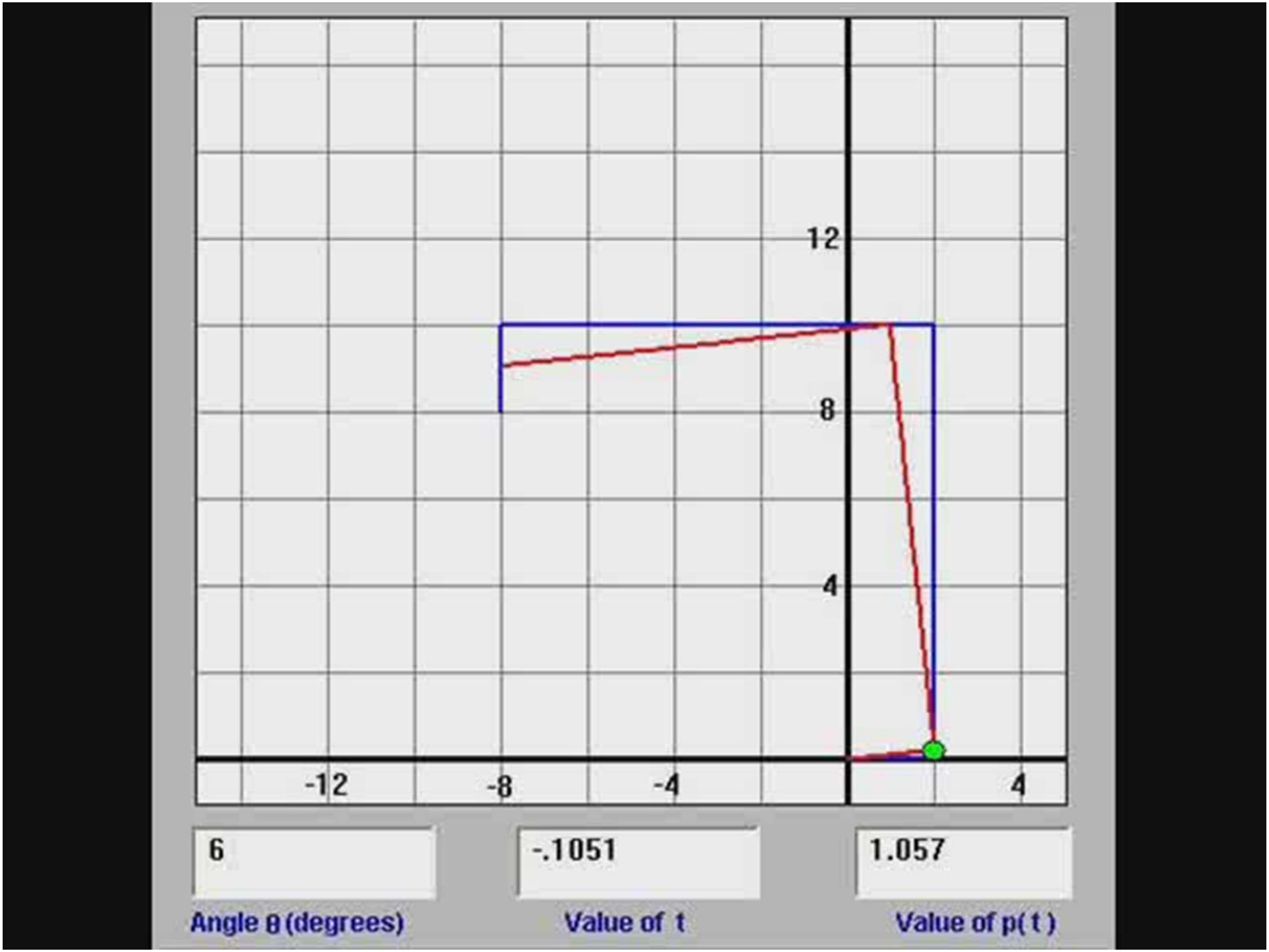
- Add a line from start to second leg of path
- Note the green angle, θ
- Add another edge perpendicular to first
- Repeat
- All green angles are equal
- Varying θ changes the end point of the secondary path
- Want paths to end at same point



Lill's Theorem

If the primary and secondary paths end on the same point, then $x = -\tan\theta$ is a root of the polynomial





12
8
4
-12
-8
-4
4
6
-.1051
1.057
Angle θ (degrees)
Value of t
Value of p(t)

Understanding Lill’s Theorem

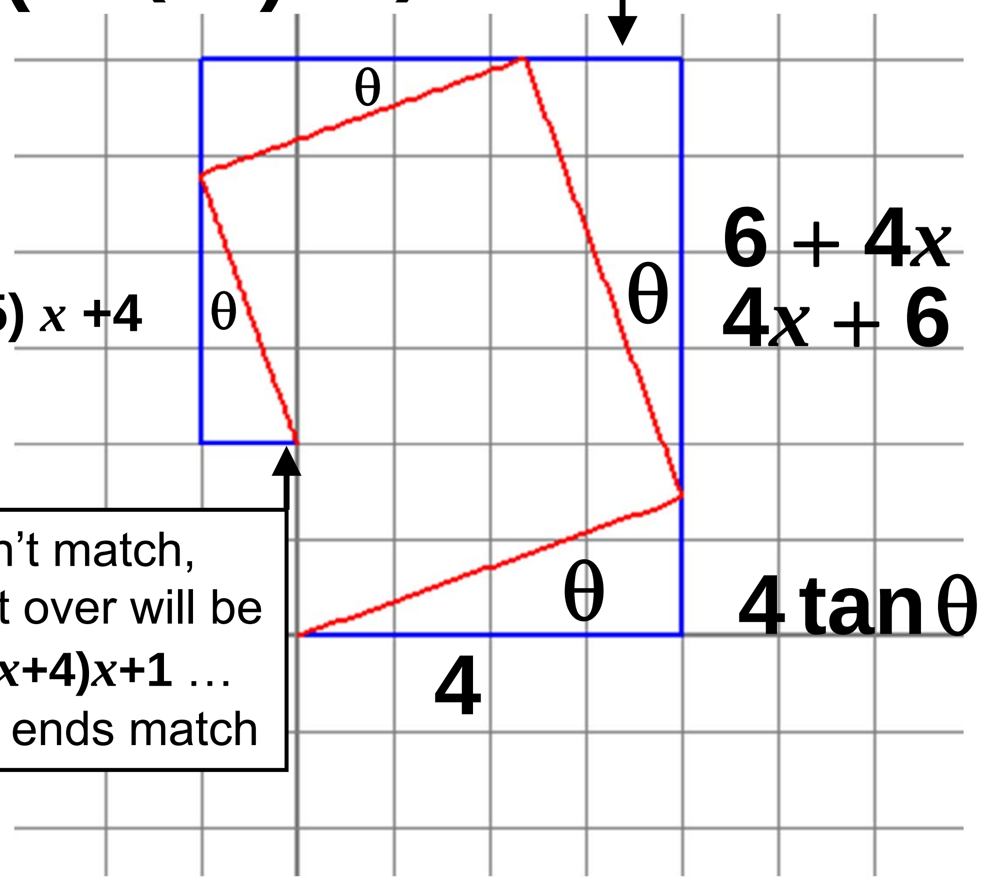
Find legs of the triangles with red hypotenuses

=

$(4x+6)x+5$ / $5+(4x+6)x$

$(4x+6)\tan\theta$

$= -(4x+6)\,x$



$((4x+6)\,x+5)\,x+4$

$6+4x$ =

$4x+6$

$4\tan\theta = -4x$

If the ends don’t match,
the blue bit left over will be
$(((4x+6)x+5)x+4)x+1$ …
This = 0 when ends match

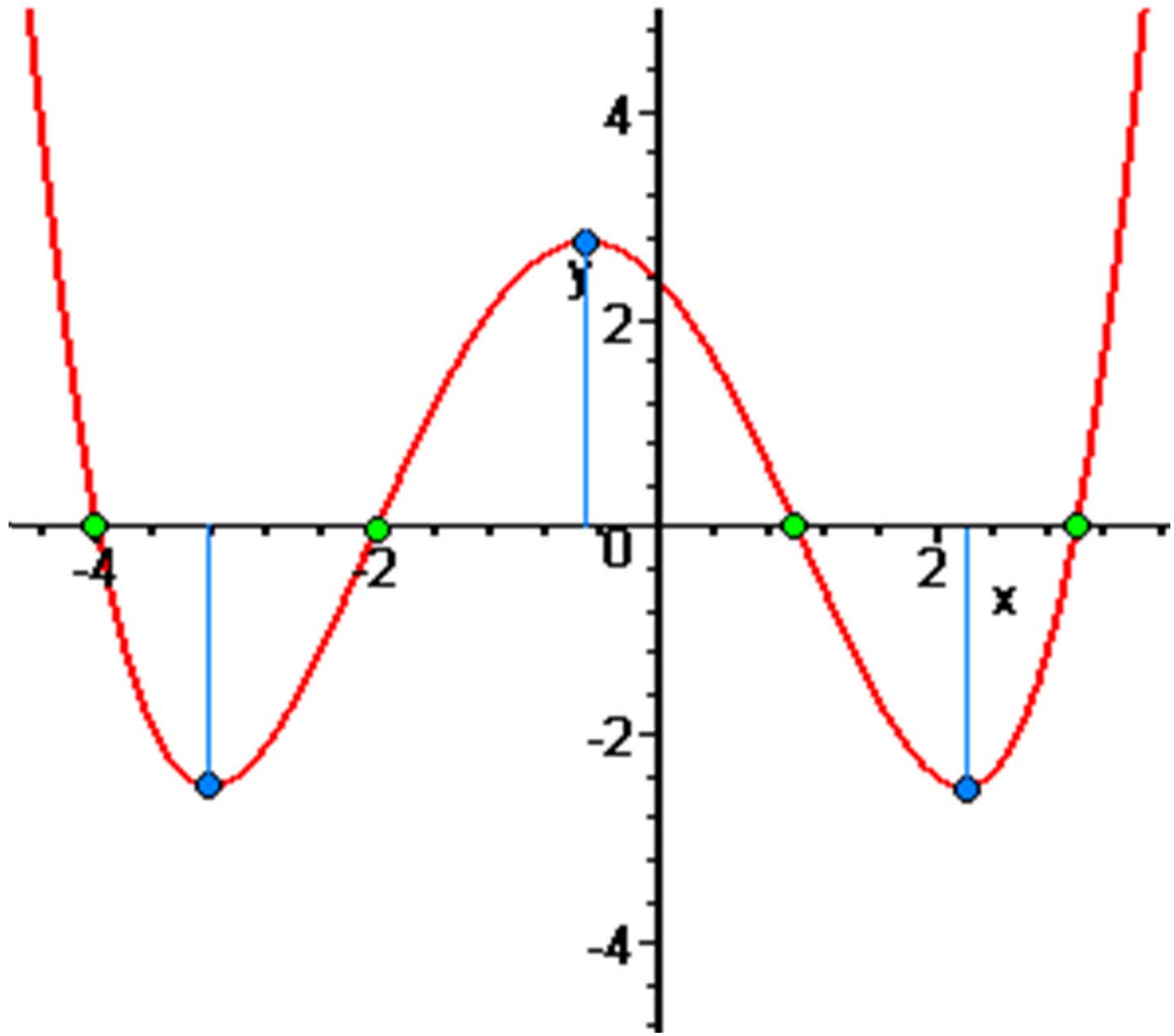
Marden’s Theorem

Marden's Theorem

- General topic: relate roots of $p(x)$ to roots of the derivative $p'(x)$
- Special case: cubic $p(x)$
- Setting: complex numbers

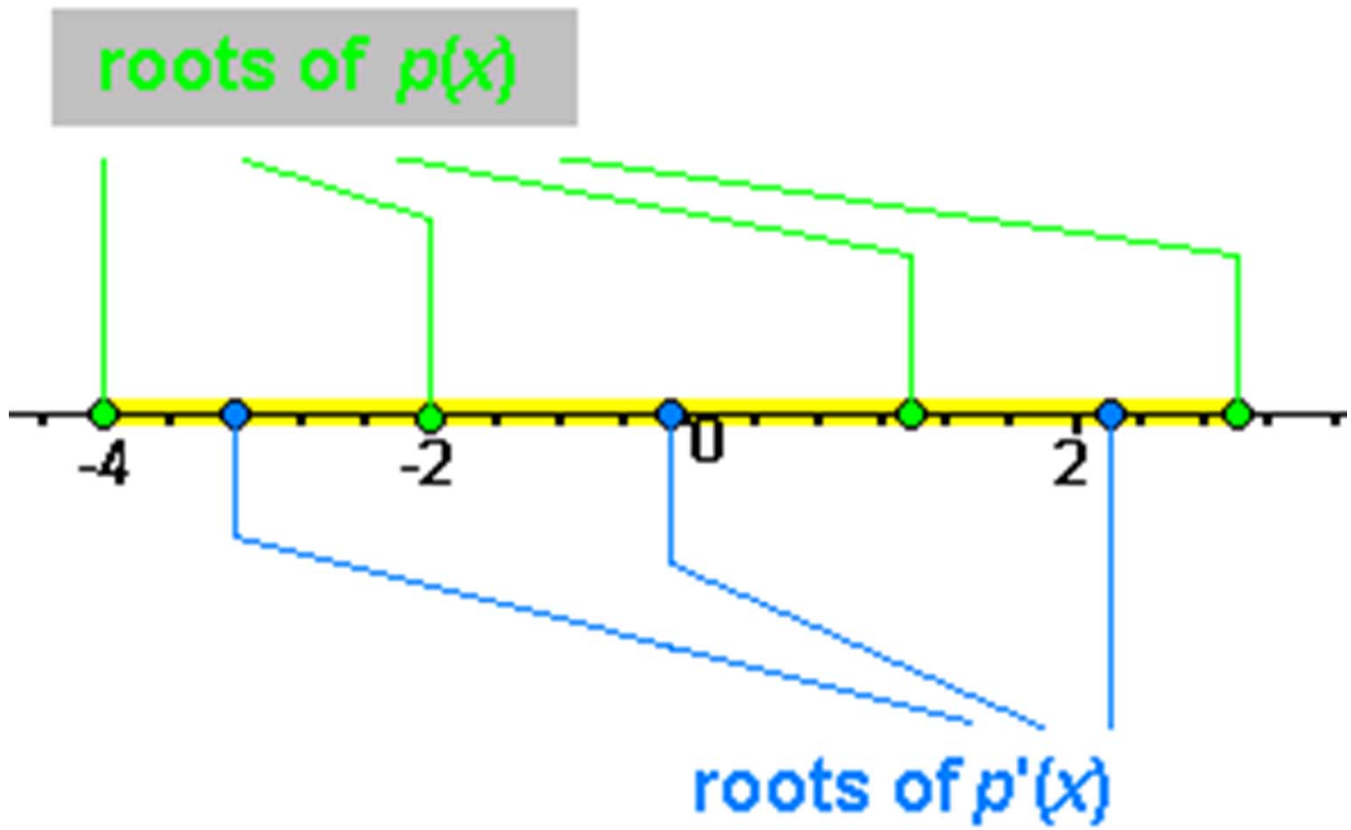
Real Polynomials with all Real Roots

Roots of $p'(x)$ interlace roots of $p(x)$

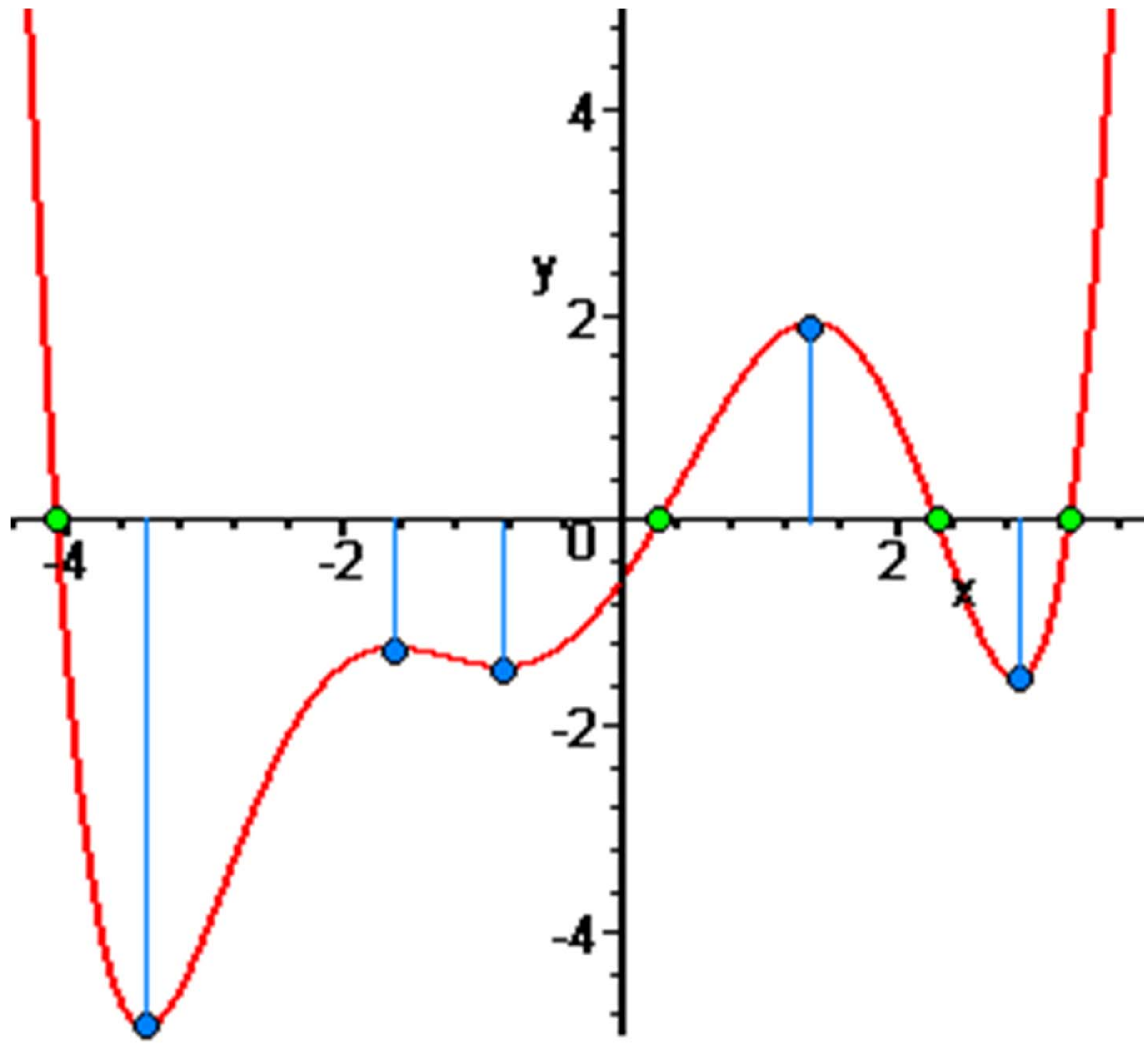


One Dimensional View

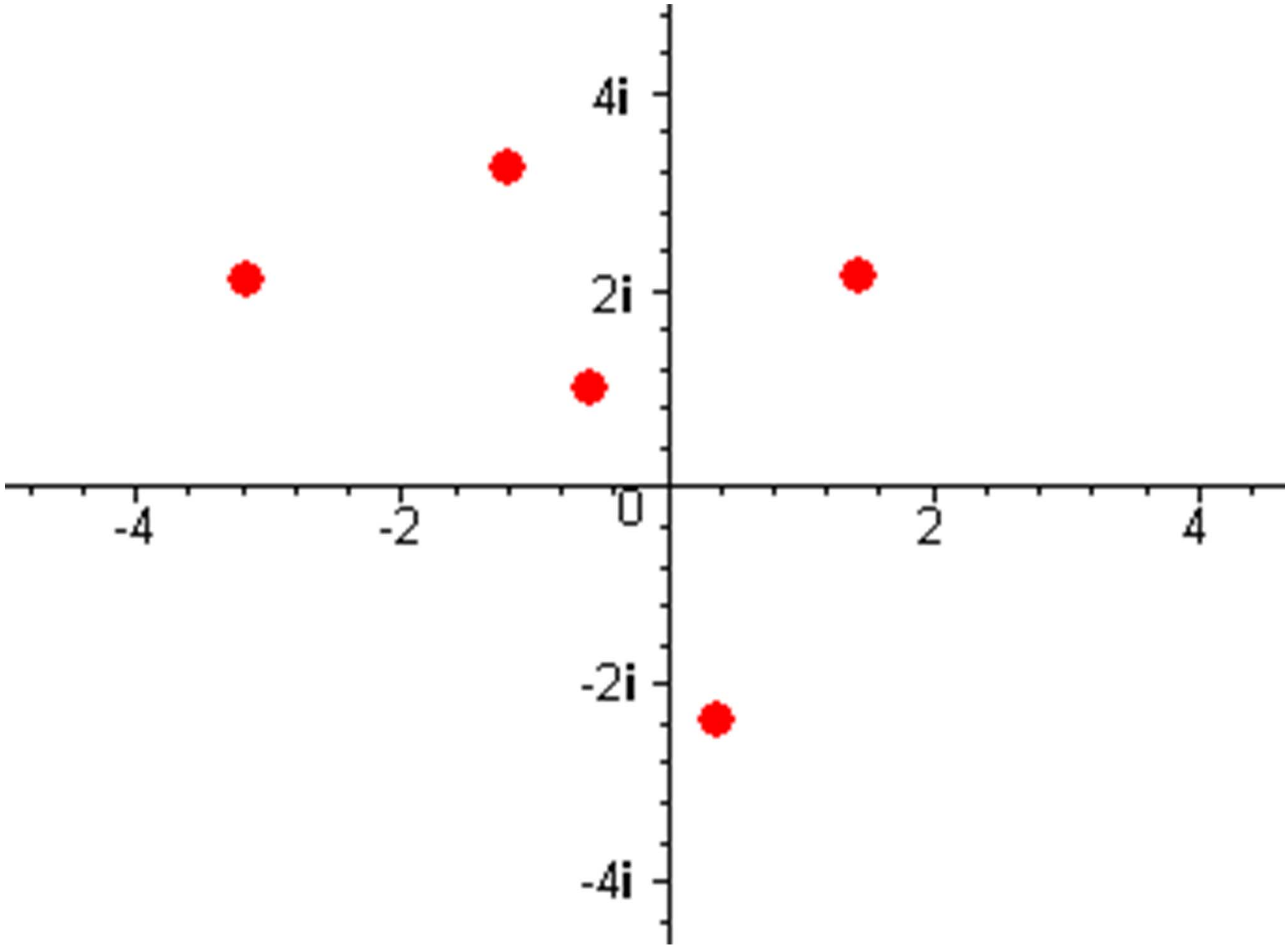
- Just view domain of $p(x)$
- Identify special points with labels



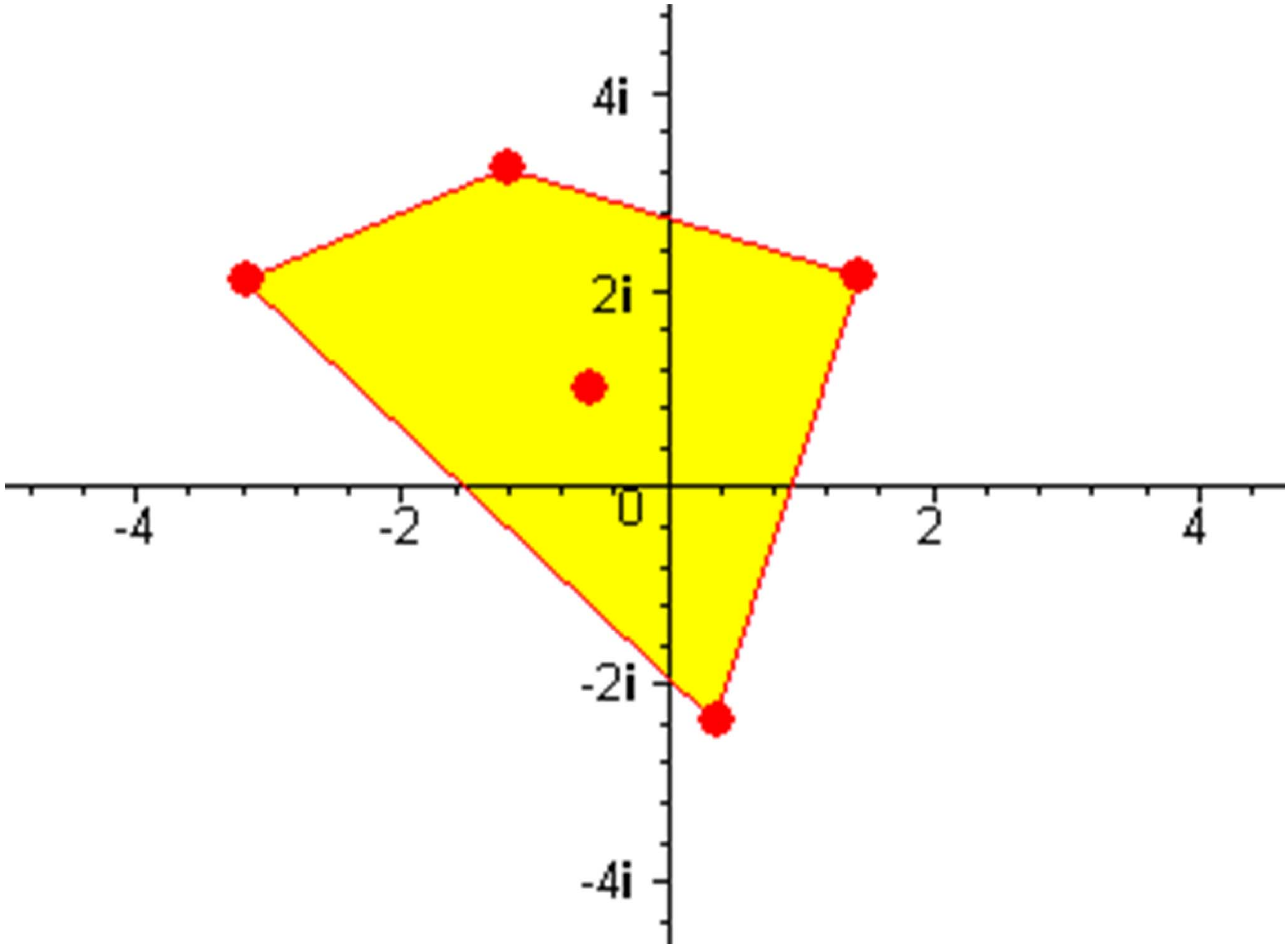
Complex roots
…?



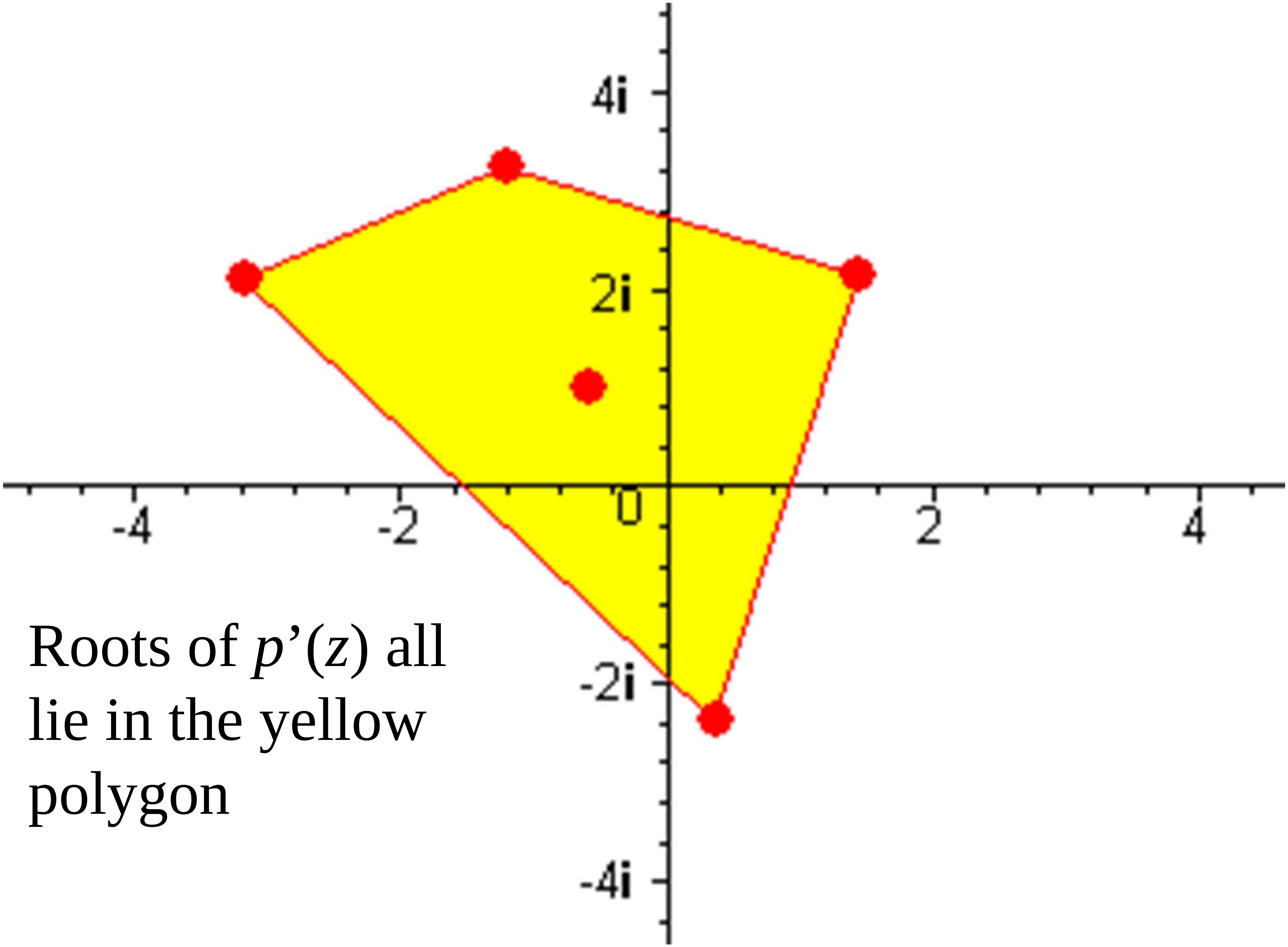
Lucas' Theorem



Lucas' Theorem



Lucas' Theorem



Roots of $p'(z)$ all lie in the yellow polygon

Marden's Theorem

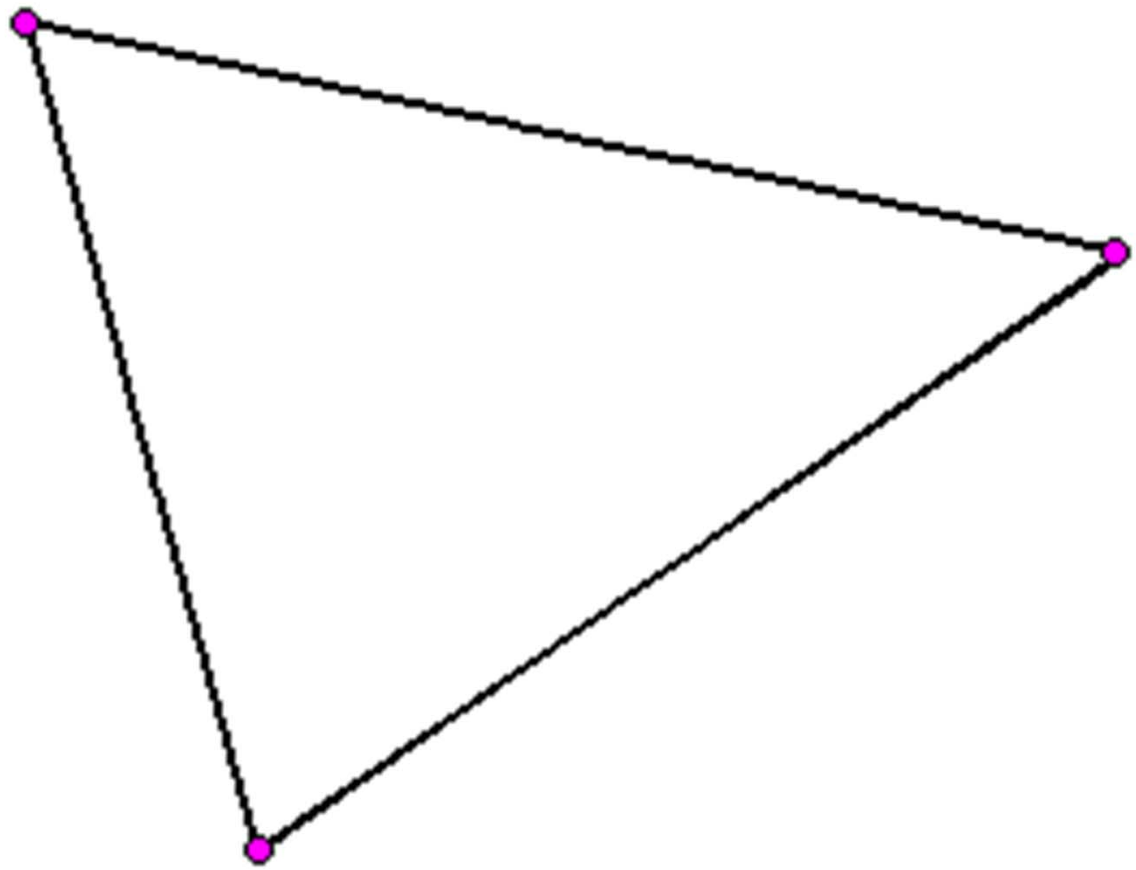
- Special case: cubic polynomial $p(z)$
- Roots are 3 noncolinear points in complex plane
- Convex hull is a triangle
- Where (exactly) are the roots of $p'(z)$?

- Show roots of $p(z)$

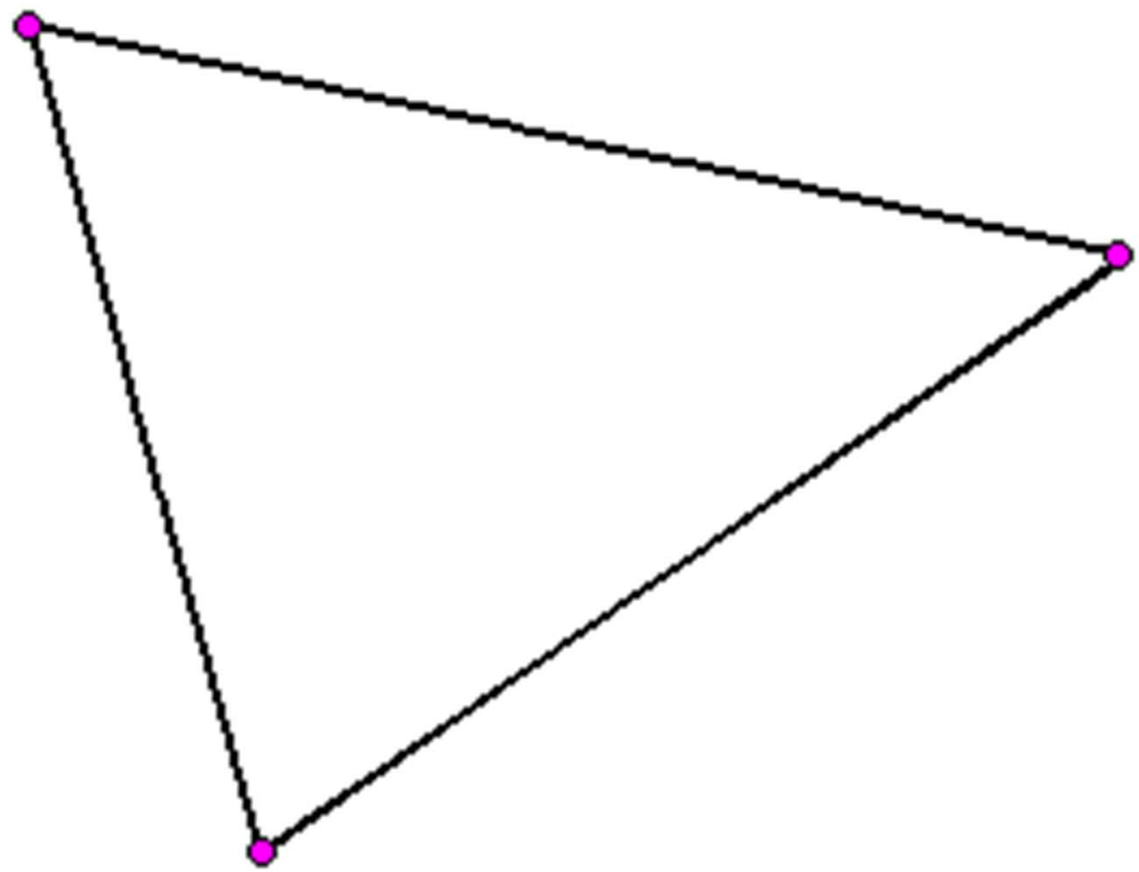
- Show roots of *p*(*z*)

- Show roots of $p(z)$
- Show triangle

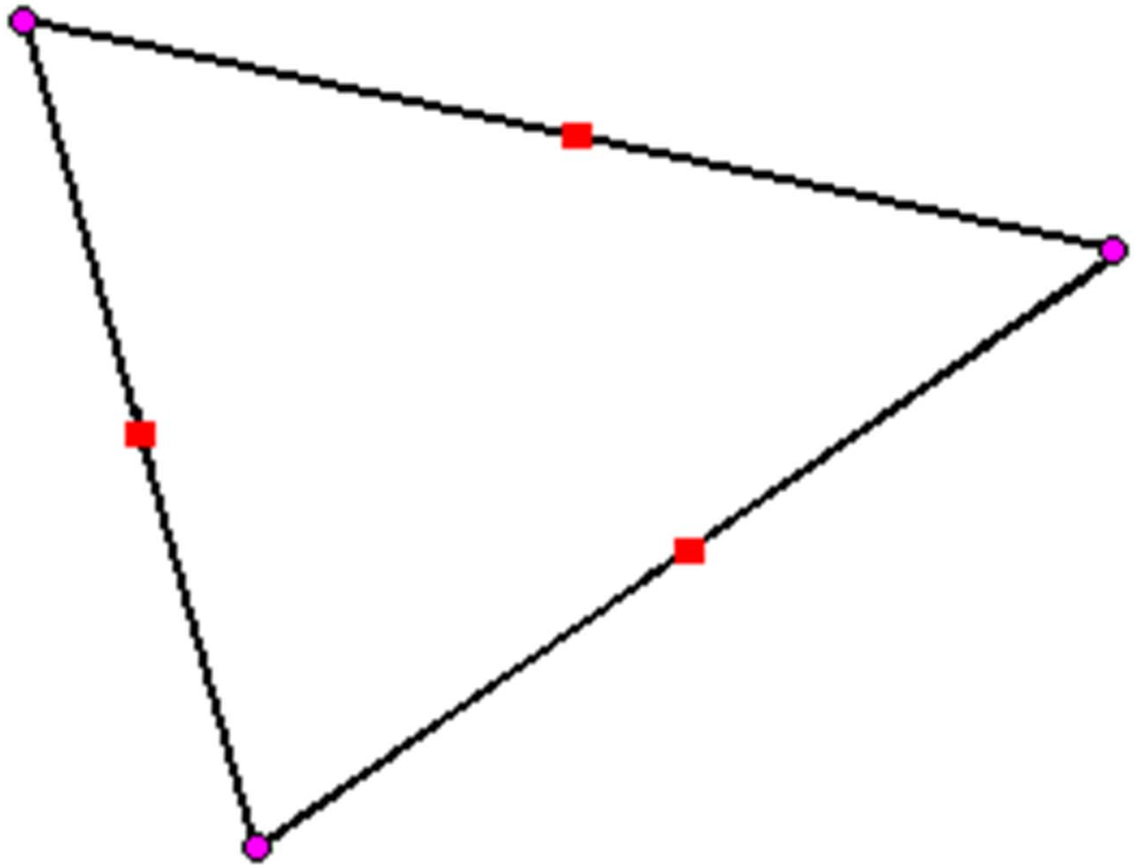
- Show roots of *p*(*z*)
- Show triangle



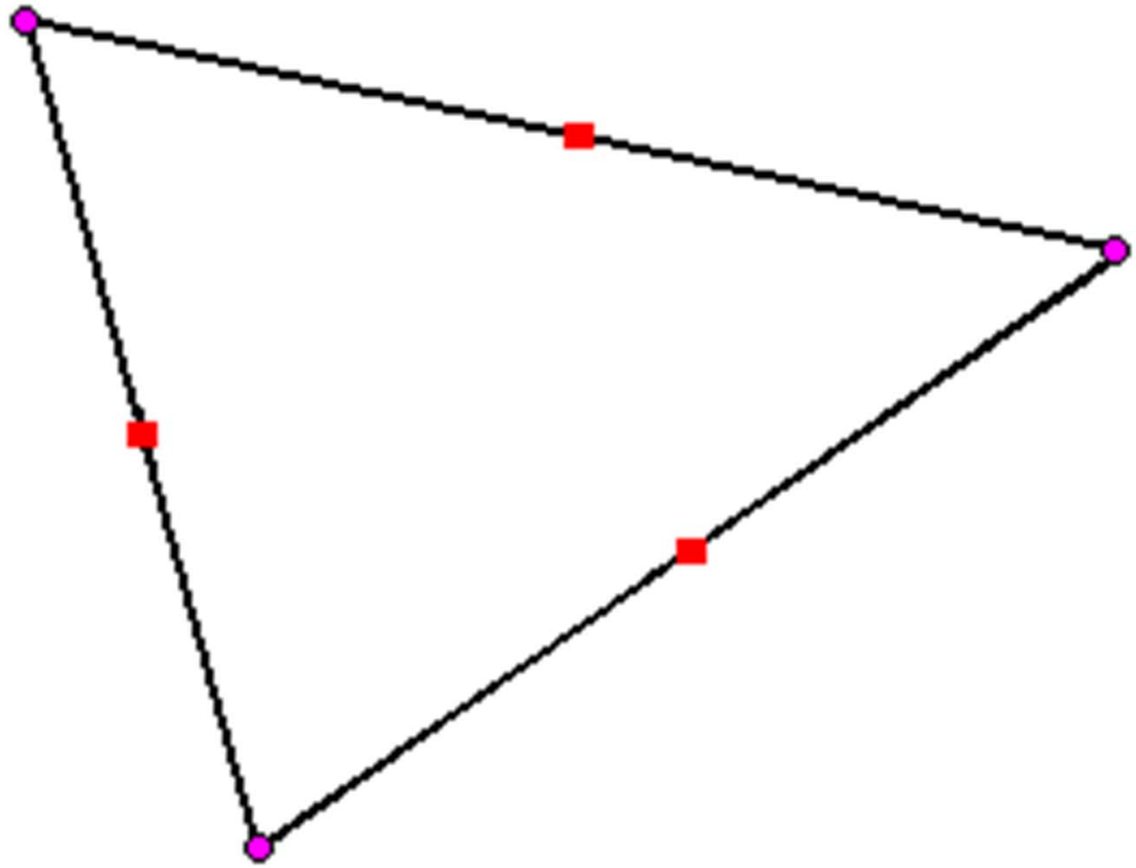
- Show roots of $p(z)$
- Show triangle
- Bisect sides



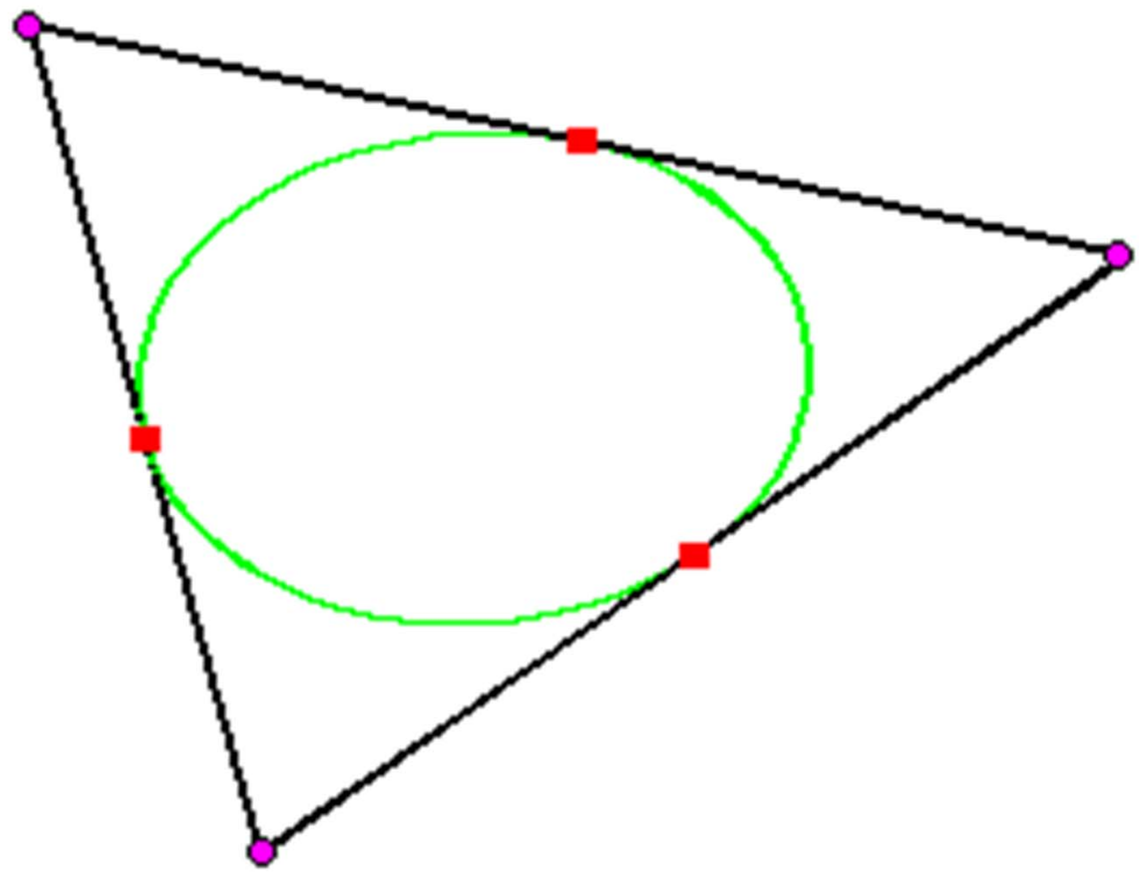
- Show roots of $p(z)$
- Show triangle
- Bisect sides



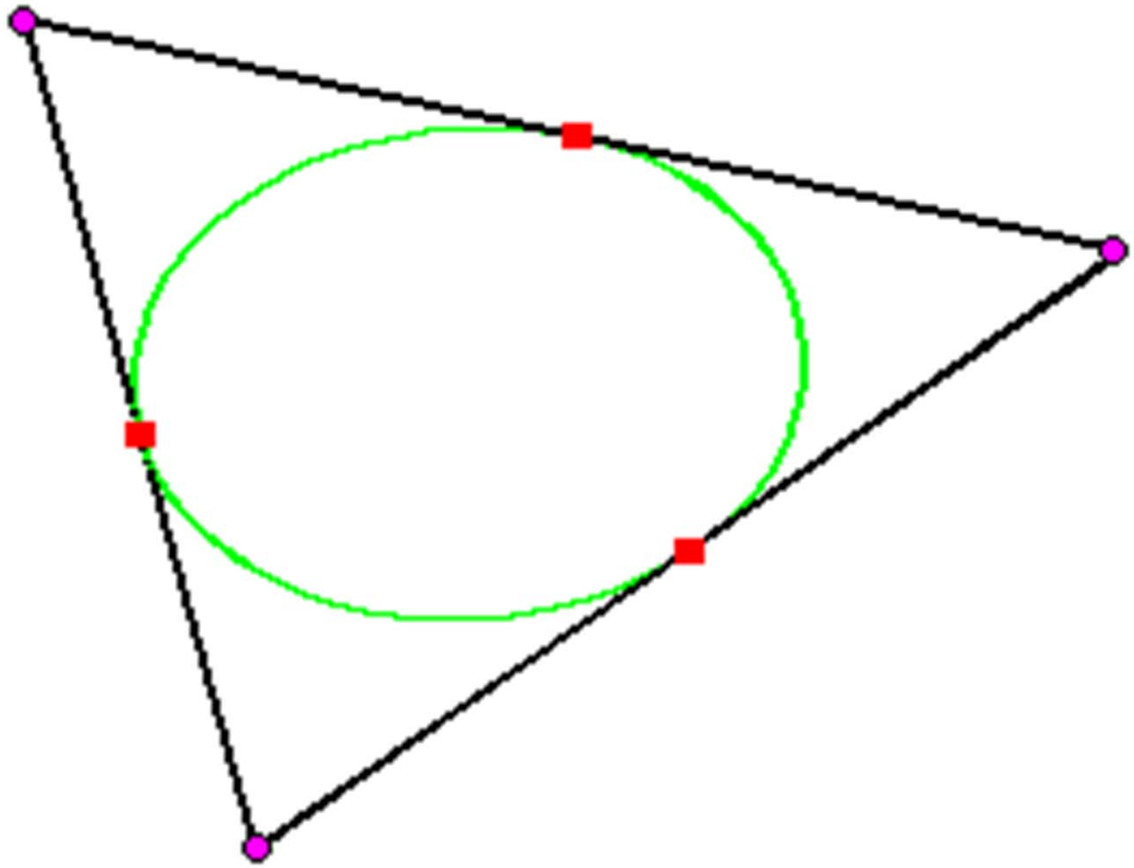
- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse



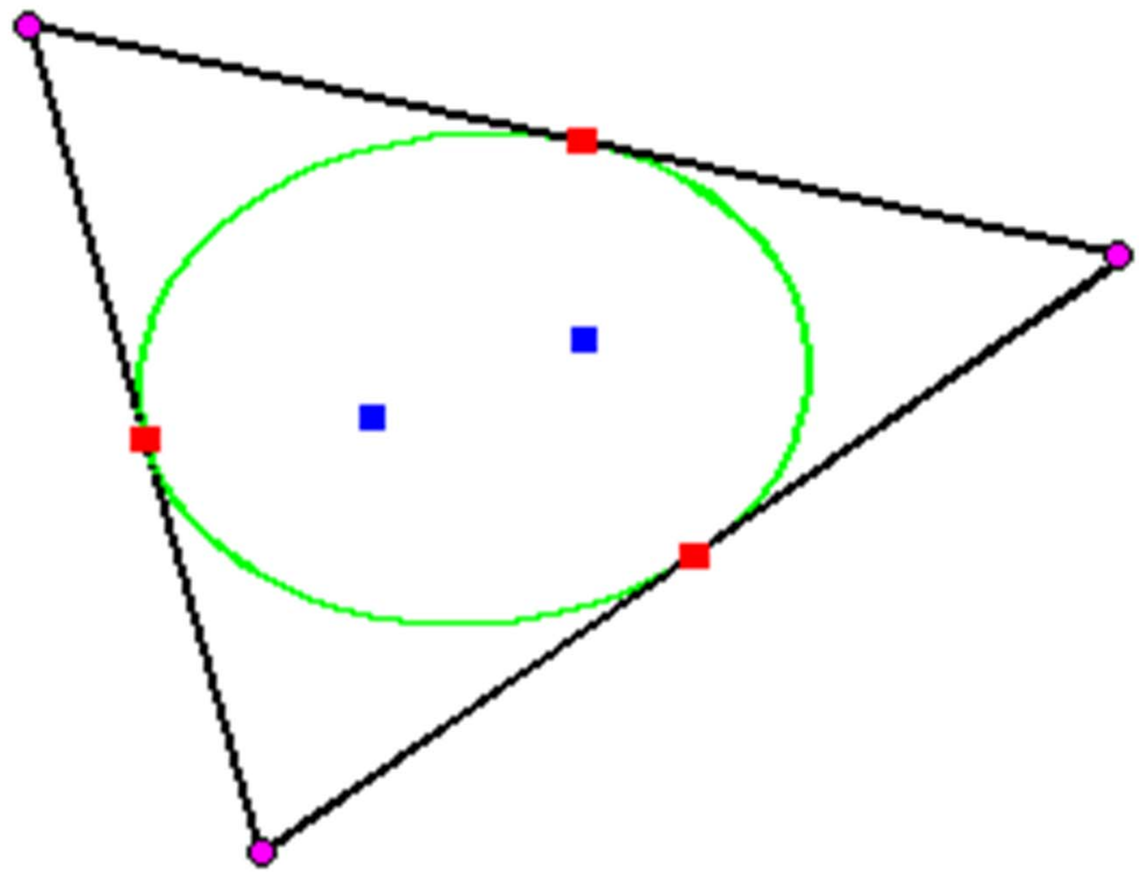
- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse



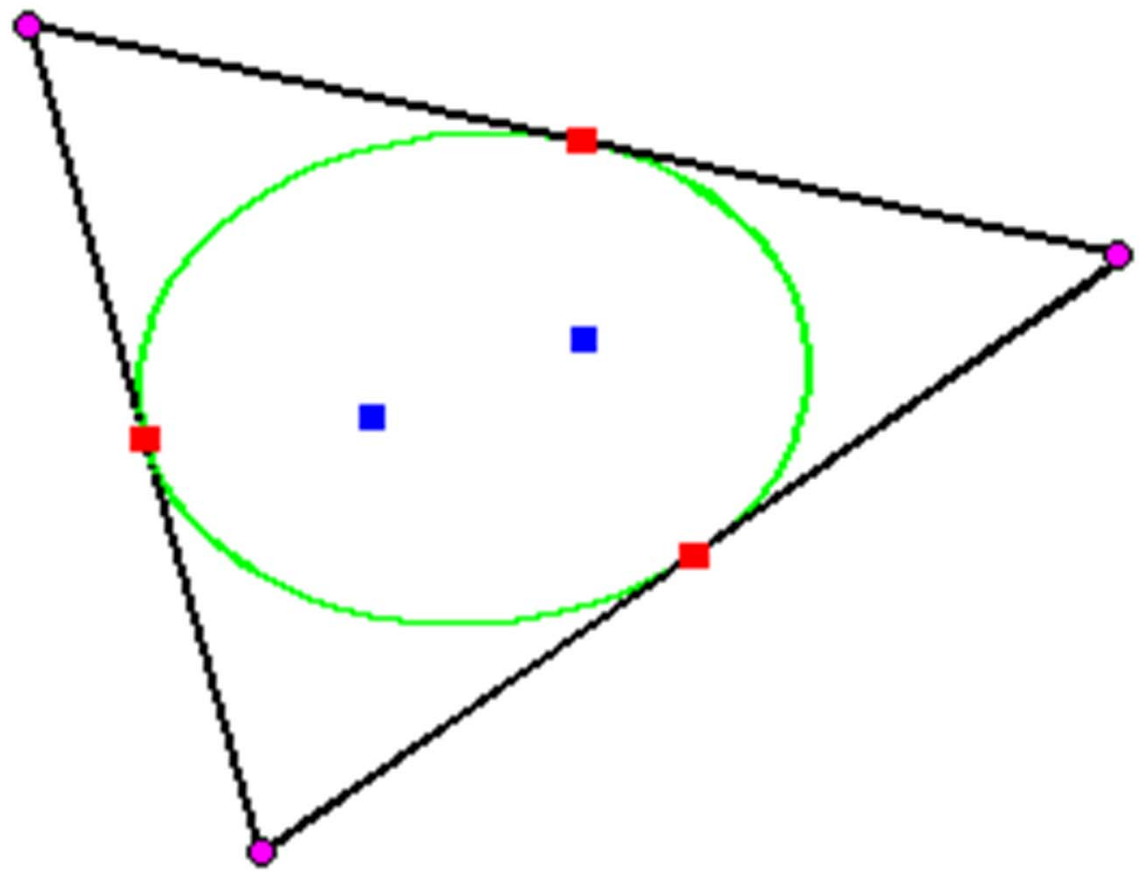
- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci



- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci



- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci
- Those are the roots of $p'(z)$



- Show roots of $p(z)$
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci
- Those are the roots of $p'(z)$
- INCREDIBLE