Pythagorean n-ples

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Pythagorean Triples

Equivalent Defs: Integer triple (a, b, c) for which

- $a^2 + b^2 = c^2$
- $\left(\frac{a}{c}, \frac{b}{c}\right)$ is rational pt on unit circle
- (a, b) is integer vector with integer length c.
- Parameterization: Take $u > v \in \mathbb{Z}$
  $(u^2 - v^2, 2uv, u^2 + v^2)$ is always a PT
- A *primitive* PT (PPT) has GCD (a, b, c)=1.
  For any PT T the *derived* PPT is $T/\mathrm{GCD}(T)$.
- Every PT is either given by the above, or is an integer multiple of a PPT derived from the above

Pythagorean 4-ples

- Terminology: Pythagorean 4-tuples, or 4-ple, are denoted as P4Ts.
- Def: Integer 4-ple (a, b, c, d) for which $a^2 + b^2 + c^2 = d^2$
- Parameterization: Take $u, v, w \in \mathbb{Z}$ $(u^2 - v^2 - w^2, 2uv, 2uw, u^2 + v^2 + w^2)$ is always a P4T
- *Primitive* P4T (PP4T) and *derived* PP4T as before
- Every P4T is either given by the above, or is an integer multiple of a PP4T derived from the above
- Generalizes to PnT's in an obvious way

Compare Parameterizations

- PT: $(u^2 - v^2, 2uv\ , u^2 + v^2)$
- P4T: $(|u^2 - v^2 - w^2|, 2uv\ , 2uw\ , u^2 + v^2 + w^2)$
- P5T:
  $(|u^2 - v^2 - w^2 - x^2|, 2uv, 2uw, 2ux, u^2 + v^2 + w^2 + x^2)$

- PnT: $(a_1, a_2, \cdots, a_n)$ where
  $u_1, u_2, \cdots, u_{n-1} \in \mathbb{N}$, and
  $a_1 = \left|u_1^2 - \sum_{k=2}^{n-1} u_k^2\right|$
  $a_k = 2u_1u_k$ for $k = 2 \cdots n - 1$
  $a_n = u_1^2 + \sum_{k=2}^{n-1} u_k^2$

Application

- Find an integer 4-vector with integer length
- Take $(u_1, u_2, u_3, u_4) = (4,2,2,1)$
- $|{u_1}^2 - {u_2}^2 - {u_3}^2 - {u_4}^2| = 16 - 4 - 4 - 1 = 7$
- $2u_1u_k$ terms are 16, 16, 8
- ${u_1}^2 + {u_2}^2 + {u_3}^2 + {u_4}^2 = 16 + 4 + 4 + 1 = 25$

- So $(7,16,16,8,25)$ is a P5T and $(7,16,16,8)$ is an integer vector with length

$$25 = \sqrt{49 + 256 + 256 + 64}$$

Derivation

- Inspired by Manjul Bhargava’s 2011 Hedricks Lecture derivation for PTs
- Characterizes rational points on unit circle as intersections of the circle with lines
- If a line has rational slope, and passes through one rational point on the circle, the other intersection will also be a rational point
- This argument extends perfectly to intersections of lines with the unit sphere in n dimensions

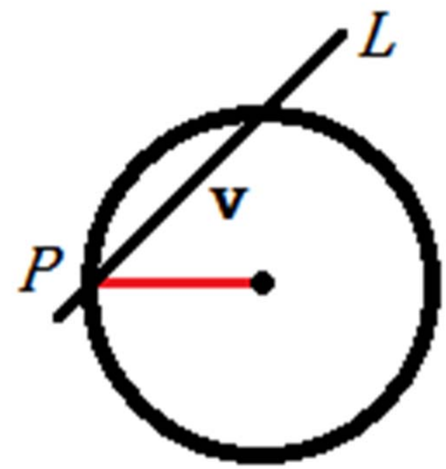
Three Dimensional Case

- If (a, b, c, d) is a P4T then $\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$ is a rational point on unit sphere in $\mathbb{R}^3$.
- Any rational point on the sphere can be expressed in the form $\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$ for some P4T (a, b, c, d)
- Consider a line through $(-1,0,0)$ parallel to a rational vector, and hence to an integer vector (u, v, w)
- Assume $u \neq 0$ so the line is not tangent to the sphere
- We find the other point of intersection

Key Equivalence

- Let L be a line through $P = (-1,0,0)$ not tangent to the sphere.
- There is a point $Q \neq P$ where L meets the sphere
- Claim: Q is a rational point iff L is parallel to a rational vector (iff parallel to an integer vector)
- (⇒) Assume Q is a rational point. Know P is a rational point. ∴ $Q - P$ is rational and $L \parallel Q - P$
- (⇐) Assume $L \parallel \mathbf{v}$ for some integer vector $\mathbf{v}$. Then $L = \{P + t\mathbf{v} \,|t \in \mathbb{R}\}$. Next slide: Intersect with sphere to find Q, show it is a rational point.

Solving for Q



- Line L: all points $\mathbf{r}(t) = P + t\mathbf{v}$
- L not tangent to sphere so $P \bullet \mathbf{v} \neq 0$
- $\mathbf{r}$ on unit sphere iff $\mathbf{r} \bullet \mathbf{r} = 1$
- $(P + t\mathbf{v}) \bullet (P + t\mathbf{v}) = 1$
- $P \bullet P + 2tP \bullet \mathbf{v} + t^2\mathbf{v} \bullet \mathbf{v} = 1$
- But $P = (-1,0,0) \Rightarrow P \bullet P = 1$
- $2tP \bullet \mathbf{v} + t^2\mathbf{v} \bullet \mathbf{v} = 0$
- $t(2P \bullet \mathbf{v} + t\mathbf{v} \bullet \mathbf{v}) = 0$
- $t = 0$ or $t = \frac{-2P\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}} \equiv t_1$
- $P = \mathbf{r}(0)$; $Q = \mathbf{r}(t_1)$

Finding Rational Points

- Line L through P parallel to the integer vector $\mathbf{v} = (u, v, w)$
- L not tangent to the sphere at P so $u \neq 0$.
- $\mathrm{Q} = \mathbf{r}(t_1) = P + t_1\mathbf{v} = (-1,0,0) + t_1(u, v, w)$
- $\mathrm{Q} = (-1,0,0) + \frac{-2P\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}(u, v, w)$
  $= (-1,0,0) + \frac{2u}{\mathbf{v}\bullet\mathbf{v}}(u, v, w)$
- $\mathrm{Q} = \frac{1}{\mathbf{v}\bullet\mathbf{v}}\{(-\mathbf{v} \bullet \mathbf{v}, 0,0) + (2u^2, 2uv, 2uw)\}$
- $\mathrm{Q} = \frac{1}{u^2+v^2+w^2}(u^2 - v^2 - w^2, 2uv, 2uw)$
- $(u^2 - v^2 - w^2, 2uv, 2uw, u^2 + v^2 + w^2)$ is P4T

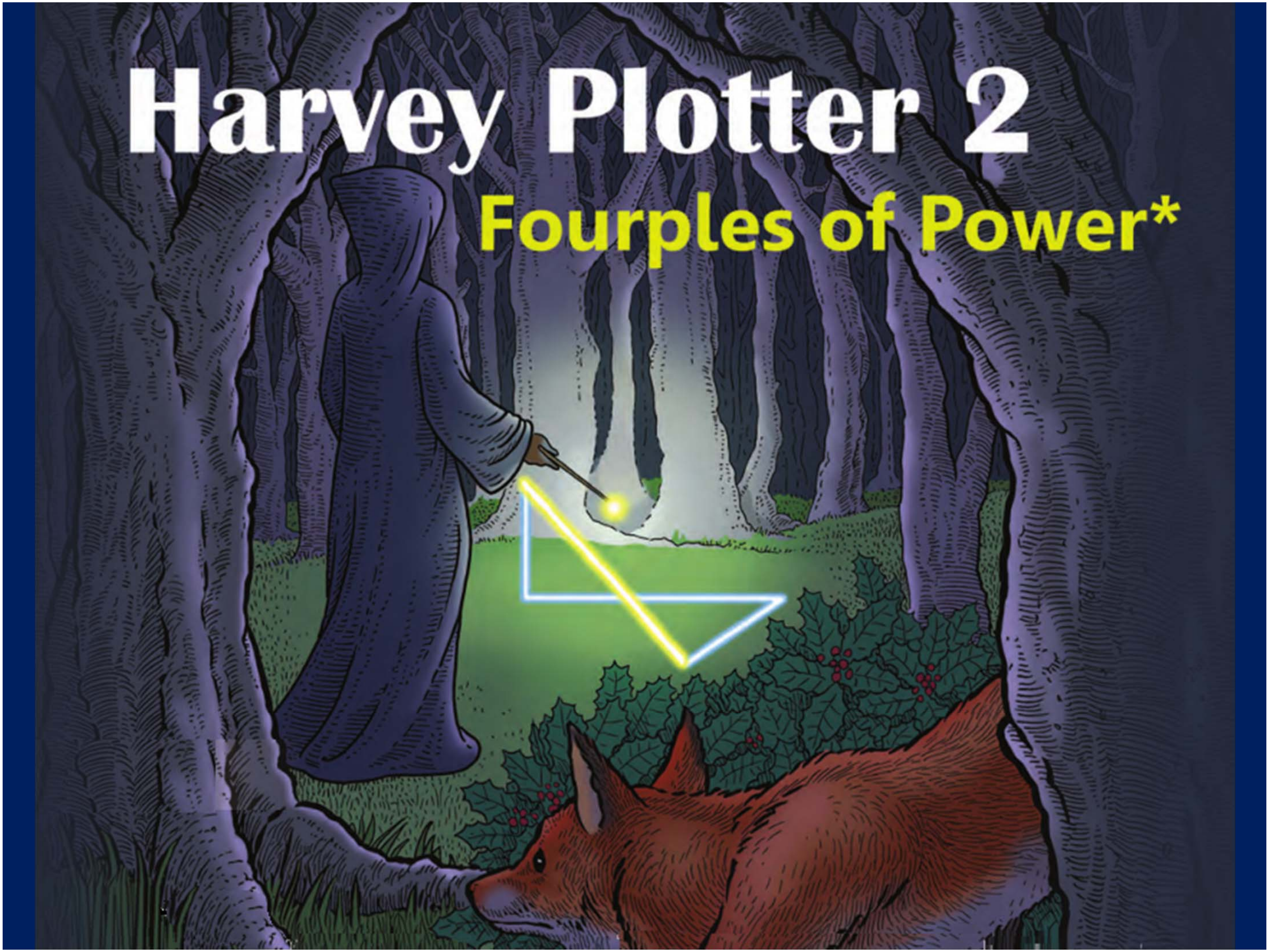
Example

- $\mathbf{v} = (u, v, w) = (1,2,3)$
- P4T is $(12,4,6,14)$
- Divide out 2 to find a PP4T $(6,2,3,7)$
- We also obtain all the integer multiples
- Also can permute first three entries $(6,3,2,7)$, $(2,6,3,7)$, $(6,2,3,7)$, etc
- Is there an efficient way to choose (u, v, w)'s so that we obtain every possible P4T without any duplication?

As some of you may know …

Bhargava's derivation of $(u^2 - v^2, 2uv\,, u^2 + v^2)$ was presented in a 2011 Math Horizons article coauthored with Nathan Carter. It parodied the then popular Harry Potter books/movies. We called it *Harvey Plotter and the Circle of Irrationality.*

Today's presentation cries out for a sequel …..


Harvey Plotter 2
Fourples of Power*

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