Generalizing a Mysterious Pattern

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(check zoom text messages for link)

Joint work will Mel Currie

Mysterious Sequence

- $a_3 = 2^3\sqrt{2 - \sqrt{2 + \sqrt{2}}}$
- $a_4 = 2^4\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$
- $a_n = 2^n\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots\sqrt{2}}}}$ (*n* radicals)
- $\lim_{n\to\infty} a_n = \pi$

Background

- Discussed in Mel's
  Mathematics Rhyme & Reason
- Arises in connection with a successive approximation scheme for area of a circle
- Closely related to Viète's 1593 infinite product for $2/\pi$
- Can we find other similar sequences and limits?

Obvious Candidates

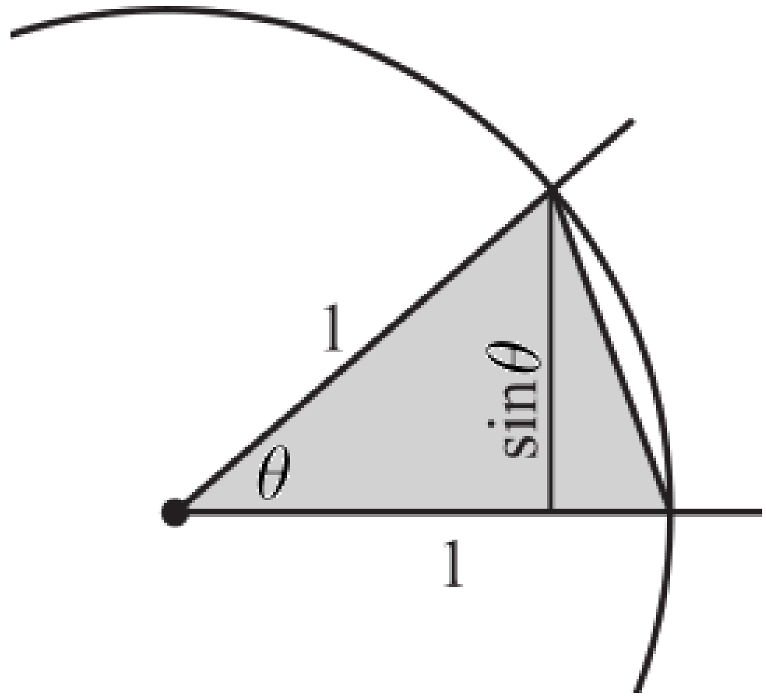
- Recall $a_n = 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}$ (*n* radicals)
- Change the 2's in the radicals?
- Change the factor 2^n ?
- Use cube roots? *n*th roots?
- Use some other functions than radicals?
- Function iteration: Let $f(t) = \sqrt{2+t}$ so

$$a_n = 2^n\sqrt{2-f\left(f\left(\cdots f(0)\right)\right)}$$

Change f ? Change innermost 0?

Geometric Context

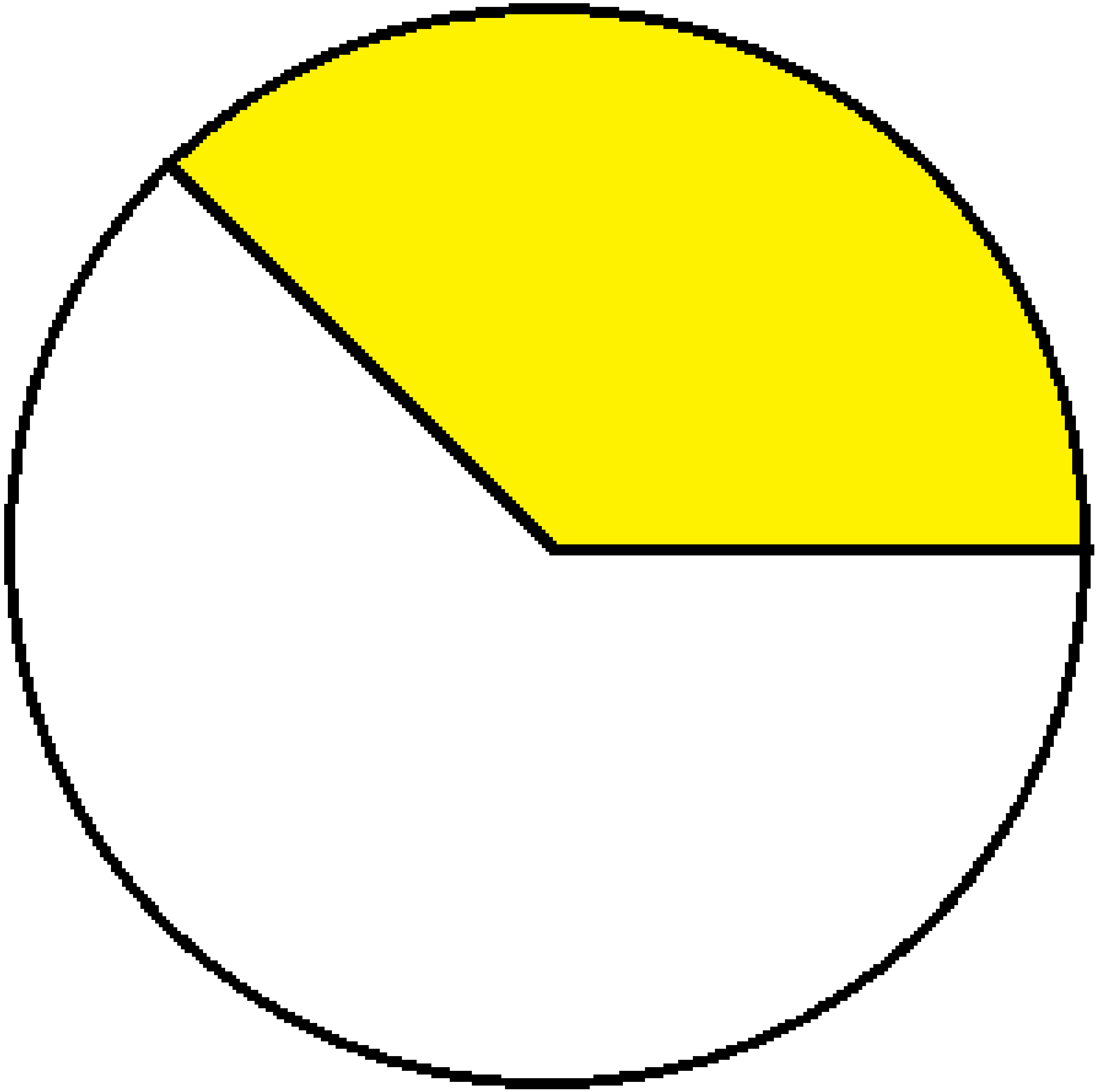
- Estimate area of unit circle = π
- Inscribed regular 2^n - gon
- Area of sector triangle is $\frac{1}{2}\sin\theta$



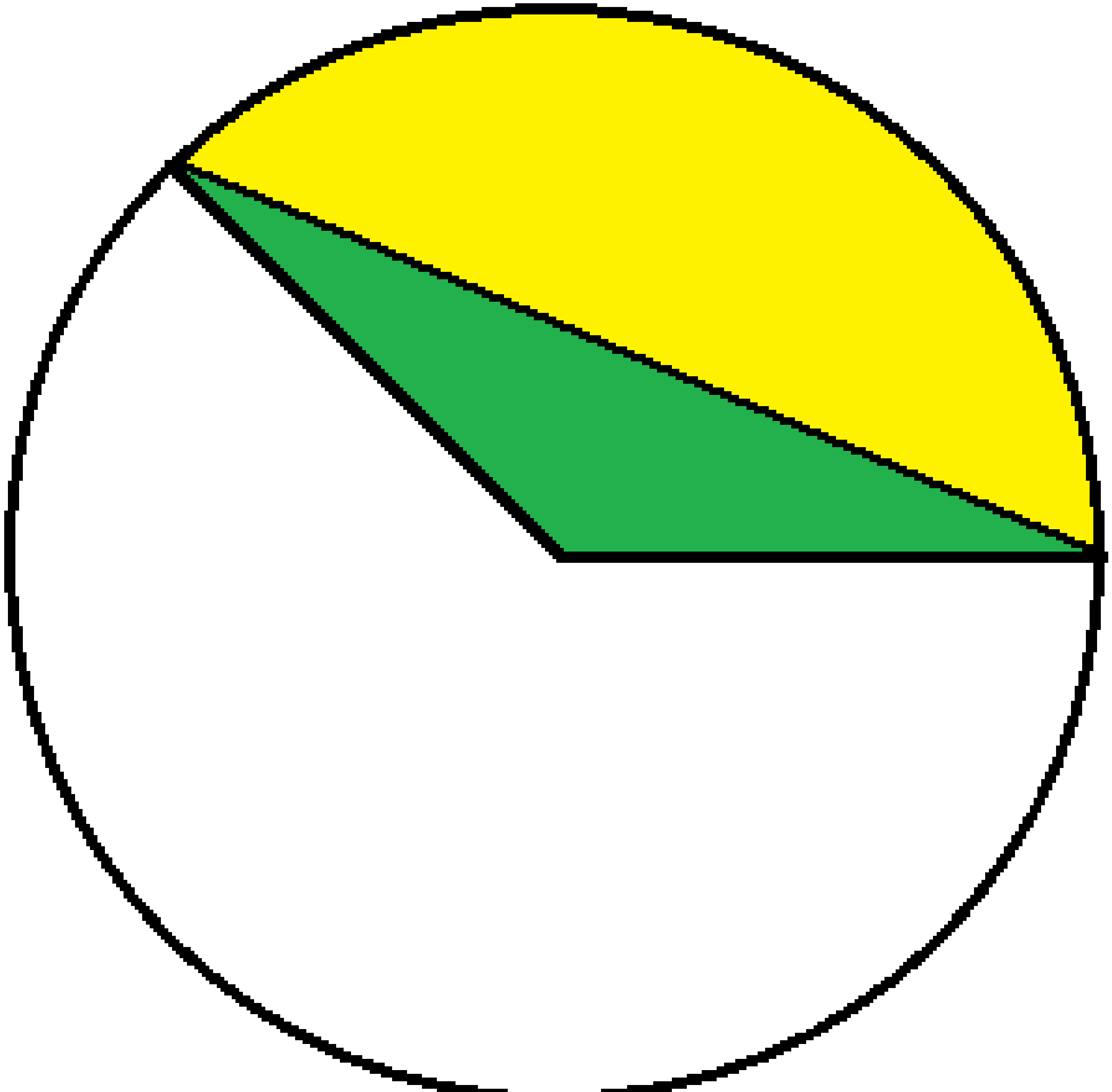
- Trig fact:

$$\frac{1}{2}\sin\frac{\theta}{2^n} = \frac{1}{4}\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots\sqrt{2 + 2\cos\theta}}}}$$

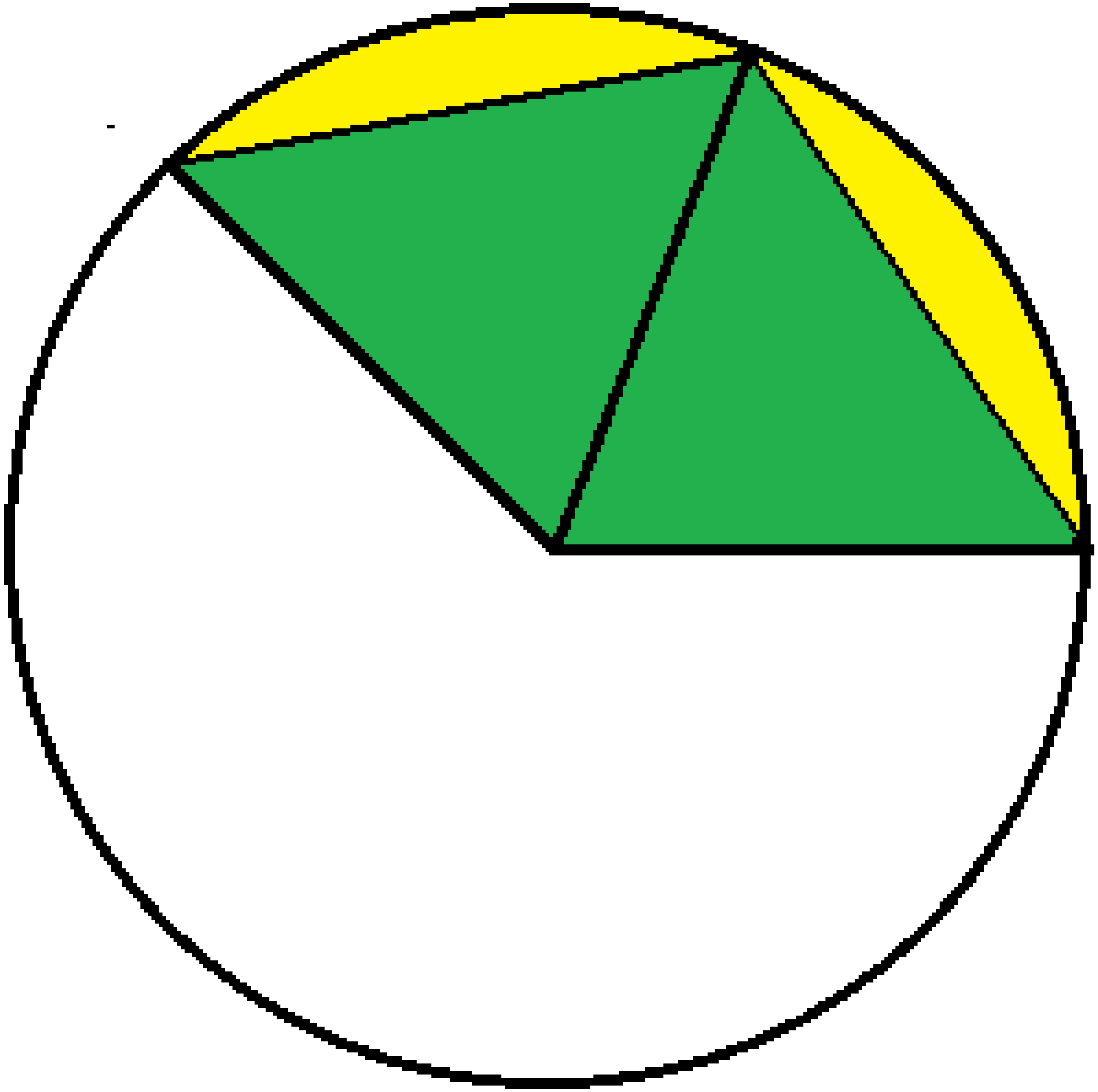
Sector Area Estimation



Sector Area Estimation



Sector Area Estimation



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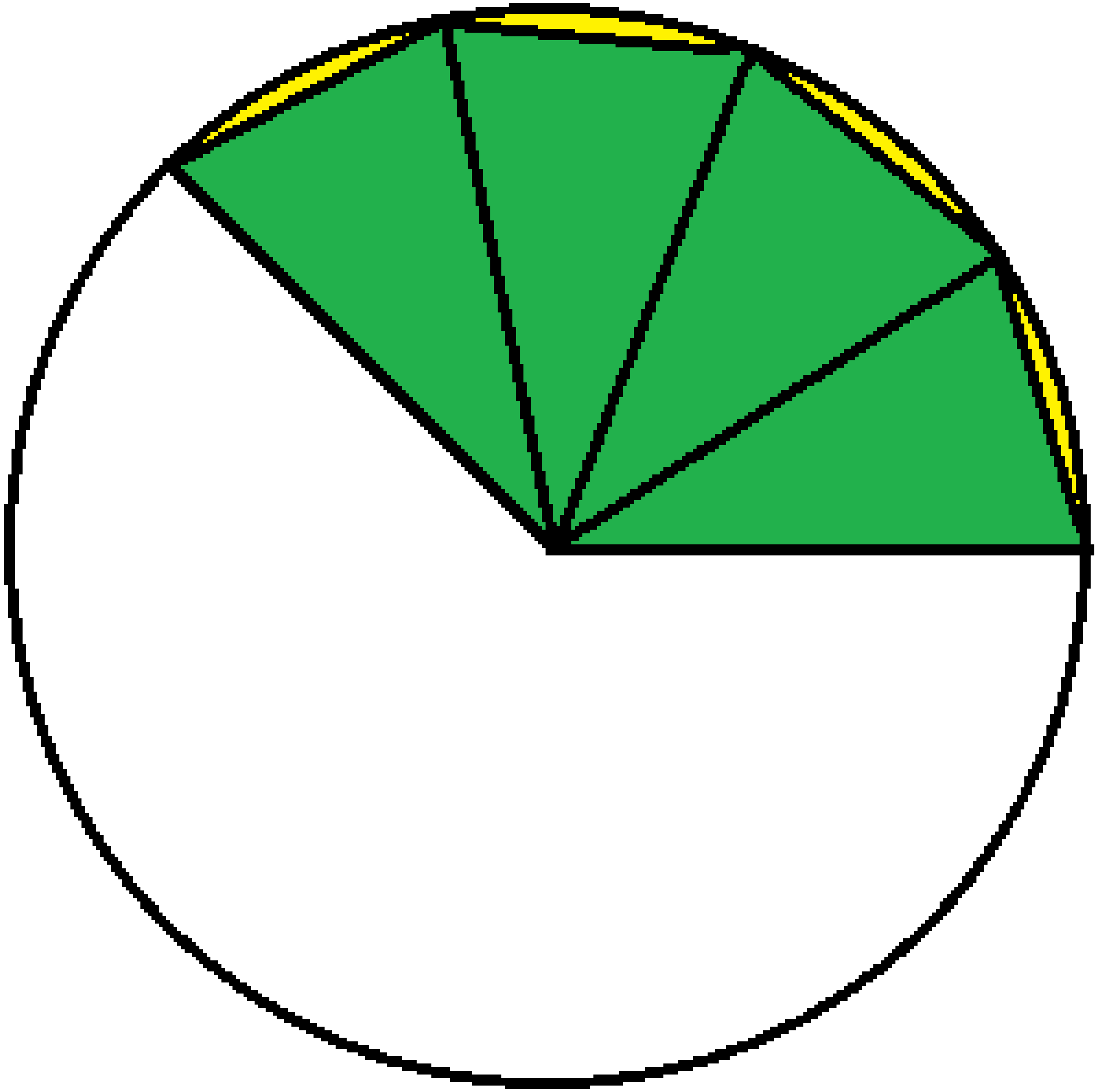


Sector Area Computation

- Compute area of a sector with central angle θ_0
- Know exact answer is $\theta_0/2$
- At the *n*th stage we have 2^n triangles each of area $\frac{1}{2}\sin(\theta_0/2^n)$
- Combined area is

$$2^n \cdot \frac{1}{4}\sqrt{2-\sqrt{2+\cdots+\sqrt{2+2\cos(\theta_0)}}}$$

- Since sector area = $\theta_0/2$

$$\lim_{n\to\infty} 2^n\sqrt{2-\sqrt{2+\cdots+\sqrt{2+2\cos(\theta_0)}}} = 2\theta_0$$

Consequences

- If $\theta_0 = \frac{\pi}{3}$ $\lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \cdots + \sqrt{3}}} = 2\pi/3$

- If $\theta_0 = \frac{2\pi}{5}$ $\lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \cdots + \sqrt{1 + \phi}}} = 4\pi/5$

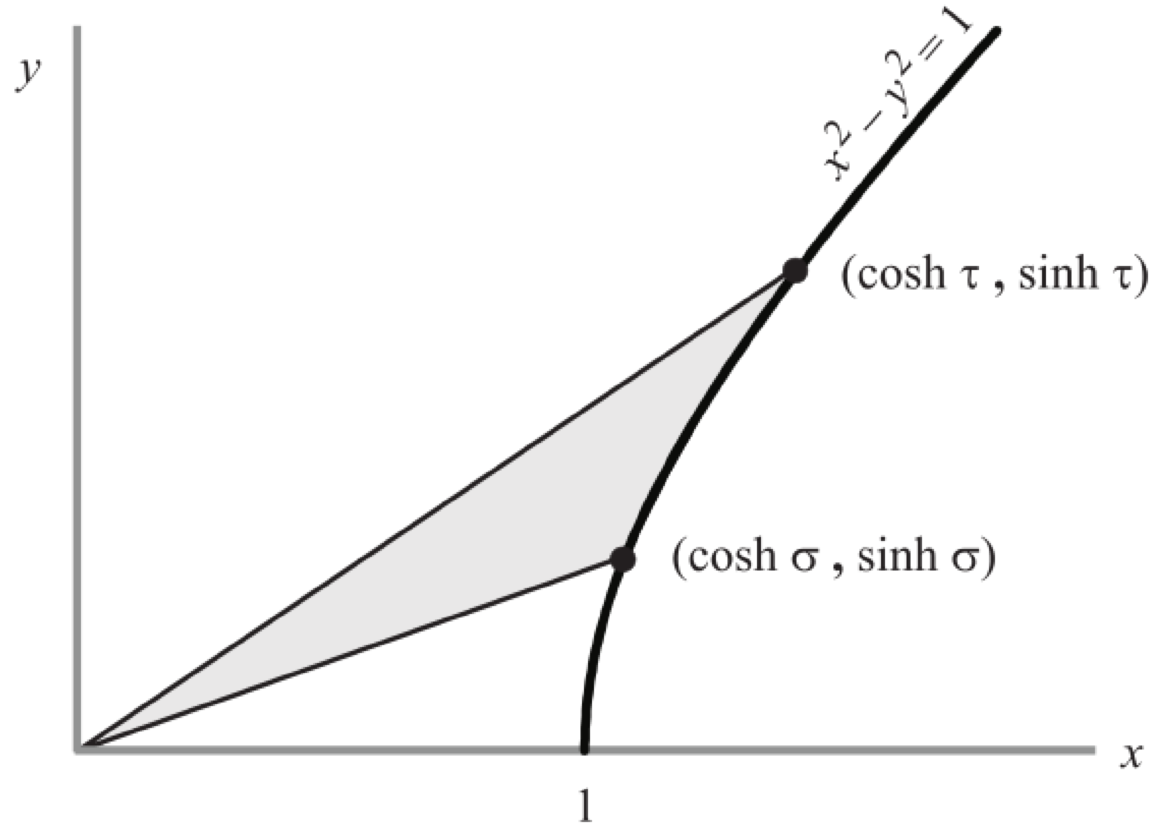
  $\left(\phi = \frac{1+\sqrt{5}}{2}\right)$

- In general for $0 \leq w < 4$

$$\lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{w}}}} = 2\cos^{-1}\left(\frac{w-2}{2}\right)$$

Hyperbolic Analog

- Estimate area of hyperbolic sector
- Hyperbolic trig replaces regular trig



- In general for $w > 4$

$$\lim_{n\to\infty} 2^n \sqrt{-2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{w}}}} = 2\cosh^{-1}\left(\frac{w-2}{2}\right)$$

Function Iteration

- Recall $a_n = 2^n \sqrt{2 - f\left(f(\cdots f(0))\right)} \equiv 2^n \sqrt{2 - f^{(n)}(0)}$
- Finite limit only possible if $f^{(n)}(0) \to 2$
- Simplify by squaring: $4^n\left(2 - f^{(n)}(0)\right)$
- Suggests considering sequences of the form $c_n = \alpha^n \left|L - f^{(n)}(t_0)\right|$ where $f^{(n)}(t_0) \to L$

- $f^{(n)}(t_0)$ has a limit when f is a contraction, and then L is a fixed point.
- For c_n to have finite positive limit, necessary that $$\alpha = 1/f'(L)$$

Candidate Sequences

- $f^{(n)}(t_0) \to L, \quad f(L) = L, \;\; f'(L) = m \;\in\; (0,1)$
- Candidate Sequence: $c_n = \left|L - f^{(n)}(t_0)\right|/m^n$
- Theorem (as of Monday this week):

  IF

  - $f^{(n)}(z_0) \to L$ for all z_0 in open interval I
  - f is increasing, concave down, f'< 1 in I
  - $f''(L) \; > \; -2m(1-m)/L$
  - $t_0 \in I, \;\; t_0 \neq L$

  THEN

  c_n converges to finite positive limit

Special Case

- Suppose $f(t) = \sqrt{K+t}$ for $t > -K$
- Fixed point L easily found: $K = L^2 - L$
- Nice parameterization $f(t) = \sqrt{L^2 - L + t}$
- $f'(L) = \frac{1}{2L}$
- Candidate sequence: $c_n = (2L)^n \left| L - f^{(n)}(t_0) \right|$
- Define $a_n = \sqrt{c_n} = (2L)^{n/2} \sqrt{|L - f^{(n)}(t_0)|}$ as a natural analog of the original *mysterious* sequence
- Think of this as a function of L so that analysis tools might be applicable. Mel showed a_n converges to positive limit for $L > 1$ when $t_0 = L - L^2$ $(\Rightarrow t_1 = 0)$
- We only know what the limit is when $L = 2$.

The Currie Function

- Changing L to x we consider the function

$$C(x) = \lim_{n\to\infty} \ (2x)^{n/2}\sqrt{x - f^{(n-1)}(0)}$$

with $\quad f(t) = \sqrt{x^2 - x + t}$

- Mel proposed this as an object of interest at the start of the project. I was the one who dubbed it the Currie function. He has discovered several interesting properties, some of which he has proved.

Some Currie Function Properties

- Product formula for $n > 1$ (proved):

$$(2x)^{n/2}\sqrt{x - f^{(n-1)}(0)} = \frac{\sqrt{x}\sqrt{2x}^{\,n}}{\prod_{k=1}^{n-1}\sqrt{x + f^{(k)}(0)}}$$

- $C(x)$ exists for $x > 1$ (proved)
- $\frac{C(x)}{x} \to \sqrt{2}$ as $x \to \infty$ (proved?)
- $\lim_{x\to\infty} C(x+1) - C(x) = \sqrt{2}$ (conj)
- Let $x_n = 1 + \frac{A}{r^n}$ where $A > 0$ and $r > 1$. Then

$$\lim_{n\to\infty} C^2(x_{n+1}) - C^2(x_n) = \ln(r^2) \text{ (conj)}$$

- We know $C(2) = \pi$. The holy grail would be to prove that $C(n)$ is transcendental for $n \geq 2$ in $\mathbb{N}$

Another Viewpoint

- Recall $c_n = (2L)^n \left| L - f^{(n)}(t_0) \right|$ where $f(t) = \sqrt{2+t}$ and fixed point $L = 2$.
- Define $\phi(t) = 2\cos t$ and $s_{1/2}(t) = t/2$.
- Then $f = \phi \circ s_{1/2} \circ \phi^{-1}$ so …
- $f^{(n)} = \phi \circ \left(s_{1/2}\right)^n \circ \phi^{-1} = \phi \circ s_{1/2^n} \circ \phi^{-1}$
- Like diagonalizing a matrix, explicit formula for $f^{(n)}$
- Sample result with $f(t) = \sqrt{2 + \sqrt{2 + t^3 - 3t}}$

$$\lim_{n\to\infty} \left(\frac{4}{3}\right)^n \sqrt{2 - f^{(n)}(2\cos\theta_0)} = \theta_0$$

Final Comments

- Many other aspects of this work that we don’t have time to consider. (continued fractions, Möbius functions, …)
- Amazingly rich topic
- So far, we have many conjectures about the Currie function and only a few hard facts
- Thanks for listening!