Some Properties of Currie's Curious Function

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Joint work with Mel Currie
Preprint available

NOTE: *Mel didn't name the function after himself – that was my doing.*

Background

- As discussed in Mel's
  Mathematics Rhyme & Reason
  (and elsewhere) …
- Let $a_n = 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}$ (*n* radicals)
- $\lim_{n\to\infty} a_n = \pi$
- Natural question: Are there other similar sequences and limits?

Obvious Example

- Let $a_n = 3^n \sqrt{3 - \sqrt{3 + \sqrt{3 + \cdots \sqrt{3}}}}$ (*n* radicals)
- What is $\lim_{n \to \infty} a_n$?
- This sequence blows up because the argument of the outer-most radical does not go to zero
- OTOH $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}} = 3$ so maybe …
- Let $a_n = 3^n \sqrt{3 - \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}$

Less Obvious Example

- $3^n \sqrt{3 - \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}$ still blows up!
- 3^n increases too fast
- Try $A^n \sqrt{3 - \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}$ for some $A < 3$
- Blows up for A too big; goes to zero for A too small
- Mel found numerically: $A = \sqrt{6}$ is just right
- Bold leap: $A^n \sqrt{L - \sqrt{M + \sqrt{M + \cdots + \sqrt{M}}}}$ where

  $M = L(L-1)$ and $A = \sqrt{2L}$

Necessary Condition

- $A^n\sqrt{L-\sqrt{M+\sqrt{M+\ \cdots+\sqrt{M}}}}$ has a finite positive limit only if $M = L(L-1)$ and $A = \sqrt{2L}$

- This potentially gives rise to an entire family of examples to consider, one for each $L > 1$.

Currie's Curious Function

- For $L > 1$ we can formulate the sequence

$$a_n = \sqrt{2L}^n \sqrt{L - \sqrt{L(L-1) + \sqrt{L(L-1) + \cdots + \sqrt{L(L-1)}}}}$$

- We expect (hope) there is a positive finite limit, determined by value of L. Call it $C(L)$.
- Parameter L becomes variable x.
- Variation of C with x may give us an analytical tool for studying the properties of $C(x)$.

Definitions

For $x > 1$

$$a_n(x) \equiv \sqrt{2x}^n \sqrt{x - \sqrt{x(x-1) + \sqrt{x(x-1) + \cdots + \sqrt{x(x-1)}}}}$$

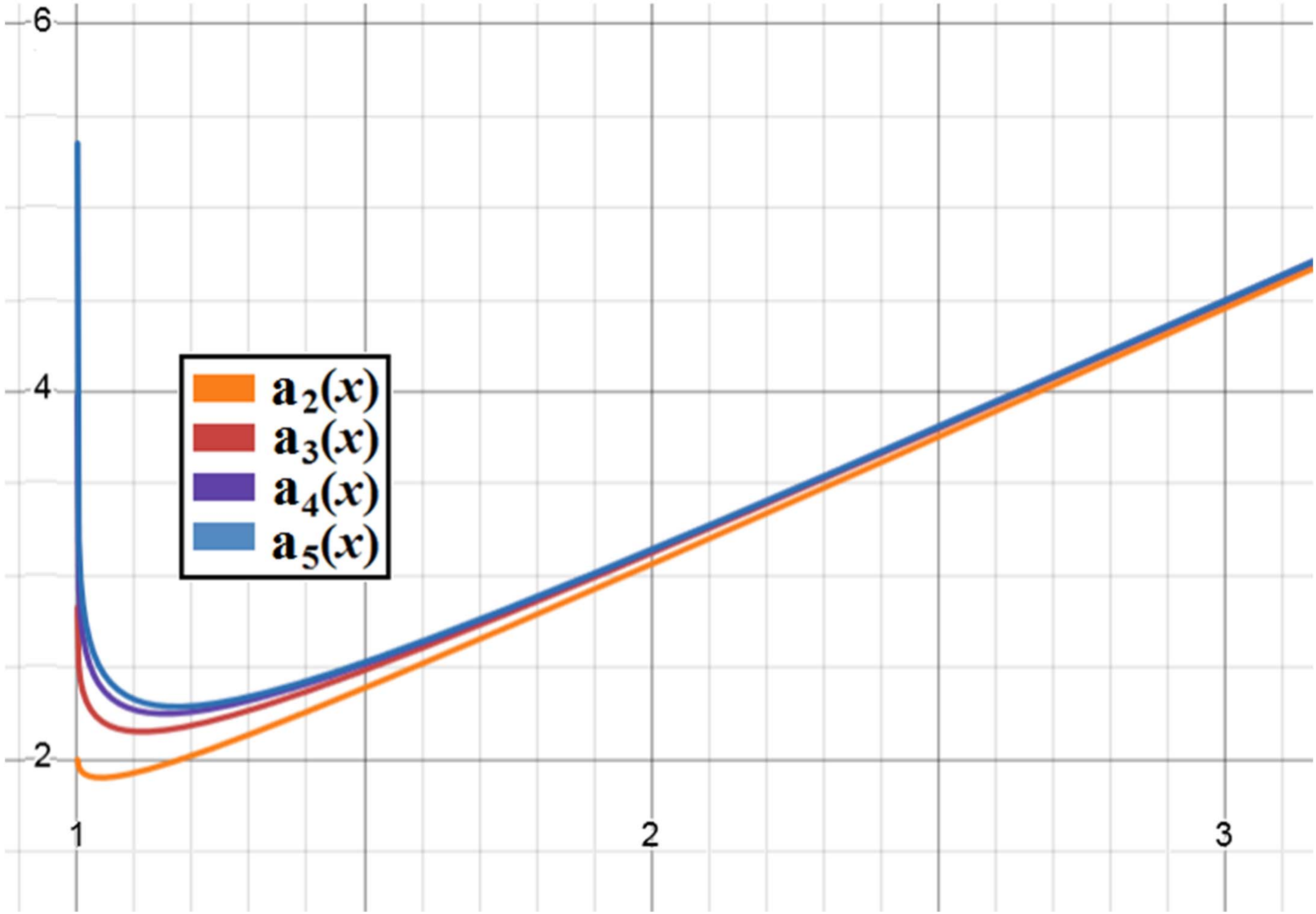
- $C(x) \equiv \lim_{n\to\infty} a_n(x)$ (where lim exists)
- Question 1: where is $C(x)$ defined?
- Note: For fixed x, $\{a_n(x)\}$ increasing sequence
  Suffices to show $a_n(x)$ is bounded

Convergence

- $C(x)$ defined for $x > 1$
  - For $x > 2$ elementary proof
  - Lemma: For any $n > 2$ $a_n(x)/x$ decreases with x
  - $\frac{a_n(x)}{x} < \frac{a_n(2)}{2} < \frac{\pi}{2} \Rightarrow a_n(x) < \frac{x\pi}{2}$ hence bounded
  - Vindicates idea of using variation with x
- For $1 < x < 2$ we do not know any similarly easy proof.
- We know 2 proofs using additional machinery:
  - Complex dynamical systems, Koenigs functions
  - Iterated real functions, Möbius approximation

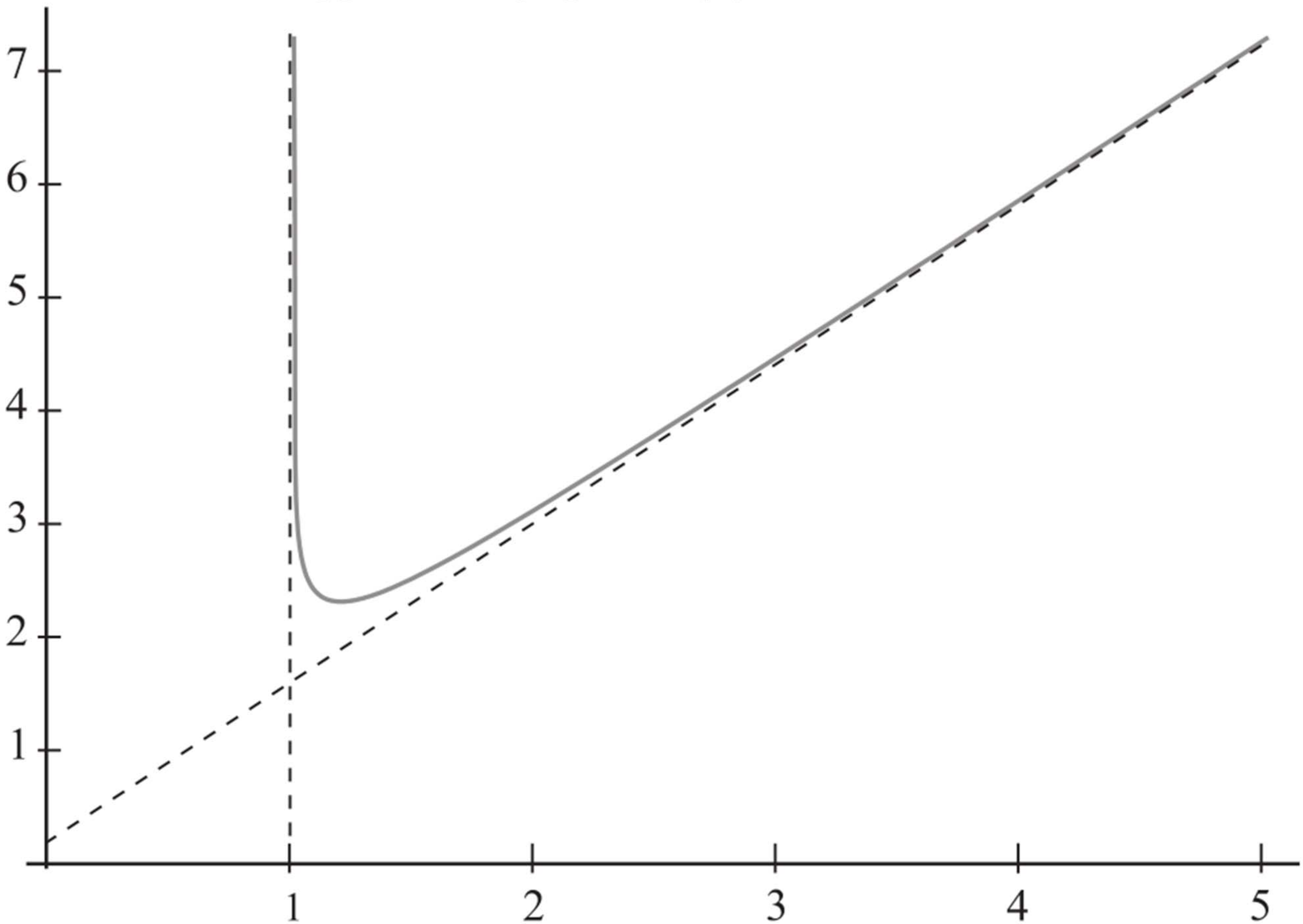
Graphical Features

- Knowing convergence justifies approximate evaluation of $C(x)$ numerically
- Mathematica provides computations to essentially arbitrary accuracy.
- Estimates of $C(x)$ to desired precision obtained by observing convergence of $a_n(x)$.
- A few sample graphs follow.



6
4
2
1
2
3
a2(x)
a3(x)
a4(x)
a5(x)

Approximate graph of $C(x)$ for $1 < x \leq 5$.



Things We Can Prove

- The linear asymptote is $L(x) = x\sqrt{2} + \sqrt{2}/8$
- This is an asymptote for $a_n(x)$ when $n \geq 2$
- If $n, x \geq 2, \;\; C(x) - a_n(x) < \frac{1}{x^{n-2}(x-1)} \leq \frac{1}{2^{n-2}}$
- $\Rightarrow \; \{a_n(x)\}$ converges uniformly on $[2, \infty)$
- $\Rightarrow C(x)$ is continuous on $[2, \infty)$
- $\lim_{x \to 1^+} C(x) = \infty$ (vertical asymptote at 1)

Conjectures

1. $\lim_{x\to 1^+} C(x)^2 + 2\ln(x-1) = 0$

2. $\lim_{x\to 1^+} C(x) - \sqrt{2|\ln(x-1)|} = 0$

3. If $A > 0$ and $r > 1$ then

$$\lim_{k\to\infty} C\left(1+\frac{A}{r^{k+1}}\right)^2 - C\left(1+\frac{A}{r^k}\right)^2 = 2\ln r$$

4. If $s > 1$ then $\lim_{x\to\infty} \frac{C(1+1/x^s)}{C(1+1/x)} = \sqrt{s}$

5. $\lim_{n\to\infty} C(\zeta(n+1))^2 - C\big(\zeta(n)\big)^2 = 2\ln 2$

Numerical Evidence Example

Conj 2: $\lim_{x \to 1^+} C(x) - \sqrt{2|\ln(x-1)|} = 0$

- Take $x_0 = 1 + e^{-625/2}$
- Less than $2 \cdot 10^{-136}$ away from 1
- $C(x_0)$ should be nearly $\sqrt{2|\ln(x_0 - 1)|} = 25$
- Ask Mathematica for $a_{400}(x_0)$ to 70 sig. figs: ans: $25.00 \cdots 0$ with 68 0's.
- $a_{400}(x_0)$, $a_{500}(x_0)$, and $a_{1000}(x_0)$ agree to 80 sig figs.
- Seems like strong evidence in favor or Conj 2.

More Conjecture Info

- Very persuasive numerical evidence for all
- No idea how to prove any of them
- We *have* proved that Conj 1 implies Conj's 2, 3, and 4
- We also have the following *compound* conjecture.
- Conjecture 6: Conj 3 implies Conj 5.
- If so, then proving Conj 1 will establish 2 -- 5

More Reasons to be Curious

- Continuity of $C(x)$ for $x \in (1,2)$
- How to show that …
  - … $C(x)$ and each $a_n(x)$ decrease to an absolute min and then increase
  - … $C(x)$ and each $a_n(x)$ is concave up on $(1, \infty)$
- Differentiability of $C(x)$ once or twice on $(1, \infty)$.
- We know the exact value of $C(x)$ only for $x = 2$. Might other exact evaluations be possible?

THE END